Filtering Data using Frequency Domain Filters

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Overview

- Intro
  - lag operator
  - Why frequency domain?
- Fourier transform
- Data as cosine waves
- Spectrum
- Filters & I(1) processes
- Band-pass filters
- HP filter

Lag operator

$$
\begin{aligned}
x_{t-1} &= Lx_t \\
x_{t-2} &= Lx_{t-1} = LLx_t = L^2 x_t \\
x_{t+1} &= L^{-1} x_t \\
\Delta x_t &= (1 - L)\, x_t
\end{aligned}
$$

Lag operator

$$
\begin{aligned}
x_t &= \rho x_{t-1} + \varepsilon_t \text{ with } |\rho| < 1 \\
x_t &= \rho L x_t + \varepsilon_t \\
(1 - \rho L)\, x_t &= \varepsilon_t \\
x_t &= \frac{\varepsilon_t}{1 - \rho L}
\end{aligned}
$$

Lag operator

$$\frac{1}{1-\rho} = 1+\rho+\rho^2+\rho^3+\cdots \text{ if } |\rho|<1$$

$$\frac{1}{1-\rho L} = 1+\rho L+\rho^2 L^2+\rho^3 L^3+\cdots \text{ if } |\rho L|<1$$

Why go to frequency domain

❶ Extract that part from the data that your model tries to explain

- e.g., business cycle frequencies

❷ Some calculations are easier in frequency domain

- e.g., auto-covariances of ARMA processes
- not the focus on this lecture

Fourier Transform

Given a sequence $\{x_j\}_{-\infty}^{\infty}$ the Fourier transform is defined as

$$F(\omega) = \sum_{j=-\infty}^{\infty} x_j e^{-i\omega j}$$

If $x_j = x_{-j}$ then

$$F(\omega) = x_0 + \sum_{j=1}^{\infty} x_j \left(e^{-i\omega j} + e^{i\omega j}\right) = x_0 + \sum_{j=1}^{\infty} 2x_j \cos(\omega j)$$

and the Fourier transform is a real-valued symmetric function.

Fourier Transform

- It is just a definition!
    - which turns out to be useful

Inverse Fourier Transform

- Given a Fourier Transform $F(\omega)$, one can back out the original sequence using

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)e^{i\omega j}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\left(\cos\omega j + i\sin\omega j\right)d\omega$$

and if $F(\omega)$ is symmetric then

$$x_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\cos\omega j\, d\omega = \frac{1}{\pi}\int_{0}^{\pi} F(\omega)\cos\omega j\, d\omega$$

- derivation is in the notes

Thinking differently about a time series

- You can take the Fourier transform of any sequence
- So you can also take it of a time series
  - take finite analogue if time series is finite

Thinking differently about a time series

- Finite Fourier transform of $\{x_t\}_{t=1}^{T}$, scaled by $\sqrt{T}$

$$\tilde{x}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} e^{-i\omega t} x_t.$$

- Let

$$\omega_j = -\pi + (j-1)2\pi/T \text{ for } j = 1, \cdots, T+1$$

Formulas

$\tilde{x}(\omega)$ can be expressed as

$$\tilde{x}(\omega) = |\tilde{x}(\omega)|\, e^{i\phi(\omega)}$$

with

$$\tilde{x}(-\omega) = |\tilde{x}(\omega)|\, e^{-i\phi(\omega)}$$

Thinking differently about a time series

- The *finite* inverse Fourier transform is given by

$$
\begin{aligned}
x_t &= \frac{1}{\sqrt{T}} \sum_{|\omega_j| \leq \pi} e^{i\omega_j t} \tilde{x}(\omega_j) \\
&= \frac{1}{\sqrt{T}} \sum_{|\omega_j| \leq \pi} |\tilde{x}(\omega)| \, e^{i\omega_j t} e^{i\phi(\omega)} \\
&= \frac{1}{\sqrt{T}} \left(\tilde{x}(0) + \sum_{0 < \omega_j \leq \pi} |\tilde{x}(\omega)| \left(e^{i(\omega_j t + \phi(\omega))} + e^{-i(\omega_j t + \phi(\omega))}
\end{aligned}
$$

Thinking differently about a time series

Using

$$e^{i\delta(\omega)} = \cos\delta(\omega) + i\sin\delta(\omega)$$

or

$$e^{i\delta(\omega)} + e^{i\delta(\omega)} = 2\cos\delta(\omega)$$

gives

$$x_t = \frac{1}{\sqrt{T}}\left(\tilde{x}(0) + 2\sum_{0<\omega_j\leq\pi}\left|\tilde{x}(\omega_j)\right|\cos(\omega_j t + \phi(\omega_j))\right)$$

Thinking differently about a time series

$$x_t = \frac{1}{\sqrt{T}} \left(\tilde{x}(0) + 2 \sum_{\omega_j \leq \pi} \left| \tilde{x}(\omega_j) \right| \cos(\omega_j t + \phi(\omega_j)) \right)$$

Simple interpretation:

- x_t : dependent variable (T observations)
- $\omega_j t$: T independent variables
- get perfect fit by choosing $\left|\tilde{x}(\omega_j)\right|$ and $\phi(\omega_j)$
- if $\left|\tilde{x}(\omega_j)\right|$ is high than that frequency is important for time variation x_t

(Informally) thinking about the variance

- What is the variance of x_t?
- Focus on the case with $\mathsf{E}[x_t] = 0$
- $\mathsf{E}\left[x_t^2\right]$ depends on $\mathsf{E}\left[\left(\sum_{\omega_j<\pi} \tilde{x}(\omega_j)\right)^2\right]$
- Fortunately, $\lim_{T\longrightarrow\infty}\mathsf{E}\left[\tilde{x}(\omega_j)\tilde{x}(\omega_i)\right] = 0$
- variance of x_t depends just on sum of the squared $\left|\tilde{x}(\omega_j)\right|$ terms (or on the integral in the limit)

Spectrum

GIven a sequence $\{\gamma_j\}_{-\infty}^{\infty}$ of autocovariances of a scalar process then the spectrum is defined as

$$S(\omega) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_j e^{-i\omega j} = \frac{1}{2\pi}\left(\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(\omega j)\right)$$

And according to the inverse

$$\gamma_0 = \int_{-\pi}^{\pi} S(\omega)\, d\omega$$

Spectrum

So spectrum is just the Fourier transform of the covariances

Spectrum of transformed series

If

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

Then

$$S_y(\omega) = b(e^{-i\omega})b(e^{i\omega})S_x(\omega) = \left| b(e^{-i\omega}) \right|^2 S_x(\omega)$$

- $|\cdot|$ is the modulus of the complex number
- Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j sequence
- For symmetric filters we have $b(e^{-i\omega}) = b(e^{i\omega})$

Examples - white noise

$$
\begin{aligned}
x_t &= \varepsilon_t \text{ and } \mathsf{E}\left[\varepsilon_t \varepsilon_{t-j}\right] = 0 \text{ for } j \neq 0 \\
S(\omega) &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} \\
&= \frac{1}{2\pi} \left(\gamma_0 + \sum_{j=1}^{\infty} 2\gamma_j \cos(\omega j) \right) \\
&= \frac{\gamma_0}{2\pi}
\end{aligned}
$$

Examples - AR(1)

$$
\begin{aligned}
y_t &= \rho y_{t-1} + x_t \\
y_t &= \frac{x_t}{1-\rho L} \\
S_y(\omega) &= \left| \frac{1}{1-\rho e^{-i\omega}} \right|^2 S_x(\omega) \\
&= \frac{1}{1-\rho e^{-i\omega}} \frac{1}{1-\rho e^{+i\omega}} S_x(\omega) \\
&= \frac{1}{1-\rho\left(e^{-i\omega}+\rho e^{+i\omega}\right)+\rho^2} S_x(\omega) \\
&= \frac{1}{1-2\rho\cos\omega+\rho^2} S_x(\omega)
\end{aligned}
$$

Examples - VAR(P)

$$
\begin{aligned}
y_t &= \sum_{j=1}^{J} A_j y_{t-j} + x_t \\
y_t &= \left(I - \sum_{j=1}^{J} A_j L^j \right)^{-1} x_t \\
S_y(\omega) &= \left(I - \sum_{j=1}^{J} A_j e^{-j\omega} \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' e^{j\omega} \right)^{-1}
\end{aligned}
$$

Examples - VAR(P)

$$
\begin{aligned}
y_t &= \sum_{j=1}^{J} A_j y_{t-j} + x_t \\
S_y(0) &= \left(I - \sum_{j=1}^{J} A_j e^{-j \times 0} \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' e^{j \times 0} \right)^{-1} \\
&= \left(I - \sum_{j=1}^{J} A_j \right)^{-1} S_x(\omega) \left(I - \sum_{j=1}^{J} A_j' \right)^{-1}
\end{aligned}
$$

This last formula is useful in calculating Heteroskedastic and

Autocorrelation Consistent (HAC) variance-covariance estimators

What is a filter?

- A filter is just a tranformation of the data
- Typically with a particular purpose
    - e.g. remove seasonality or "noise"
- Filters can be expressed as

$$
\begin{aligned}
x_t^f &= b(L)x_t \\
b(L) &= \sum_{j=-\infty}^{\infty} b_j L^j
\end{aligned}
$$

Examples of filters

$$
\begin{aligned}
b(L) &= 1 - L \Longrightarrow x_t^f = x_t - x_{t-1} \\
b(L) &= -\frac{1}{2}L^{-1} + 1 - \frac{1}{2}L
\end{aligned}
$$

I(0) and I(1) processes

- I(0) are stationary processes
- x_t is I(1) if Δx_t is stationary

I(0) and I(1) processes

$$x_t = B(L)\,\varepsilon_t,$$

where ε_t is white noise.

- If x_t is I(1), then $B(1) = \infty$
- If $B(1) < \infty$, then x_t is stationary

(same holds is ε_t is stationary)

Filters that induce stationarity

- Suppose that x_t is I(1). Thus

$$(1 - L)x_t = z_t$$

  with z_t an I(0) process.

- Filtering gives

$$x_t^f = b(L)x_t$$

- Question: When is x_t^f I(0)?

Filters that induce stationarity

Define $\bar{b}(L)$ such that

$$b(L) = (1 - L)\bar{b}(L).$$

If

$$\bar{b}(1) < \infty,$$

then $x_t^f = b(L)x_t$ is stationary even if x_t is I(1).

$$\begin{aligned} x_t^f &= b(L)x_t \\ &= (1 - L)\bar{b}(L)x_t \\ &= (1 - L)\bar{b}(L)\frac{z_t}{(1 - L)} \\ &= \bar{b}(L)z_t \end{aligned}$$

Spectrum of filtered series

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

Then

$$S_y(\omega) = b(e^{-i\omega})b(e^{i\omega})S_x(\omega) = \left|b(e^{-i\omega})\right|^2 S_x(\omega)$$

- $|\cdot|$ is the modulus of the complex number
- Note that $b(e^{-i\omega})$ is the Fourier transform of the b_j sequence
- For symmetric filters we have $b(e^{-i\omega}) = b(e^{i\omega})$

Band-pass filters

$$y_t = b(L)x_t$$

Goal:

$$S_y(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_L \leq |\omega| \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

Thus we need

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_L \leq |\omega| \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

- How to find the coefficients b_j that correspond with this?
- Since $b(e^{-i\omega})$ is a Fourier transform we can use the inverse of the Fourier transform

Coefficients of band-pass filters

Inverse of the Fourier transform for b_0:

$$
\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{i\omega j}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\omega_H}^{-\omega_L} 1\times e^{i\omega j}d\omega + \int_{\omega_L}^{\omega_H} 1\times e^{i\omega j}d\omega\right) \\
&= \frac{\omega_H - \omega_L}{\pi}
\end{aligned}
$$

Coefficients of band-pass filters

Inverse of the Fourier transform for b_j:

$$
\begin{aligned}
b_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} b(e^{-i\omega})e^{i\omega j}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-\omega_H}^{-\omega_L} 1\times e^{i\omega j}d\omega + \int_{\omega_L}^{\omega_H} 1\times e^{i\omega j}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{\omega_L}^{\omega_H}\left(e^{i\omega j}+e^{-i\omega j}\right)d\omega\right) \\
&= \frac{1}{2\pi}\int_{\omega_L}^{\omega_H} 2\cos(\omega j)d\omega \\
&= \frac{1}{\pi}\frac{1}{j}\sin\omega j]_{\omega_L}^{\omega_H} = \frac{\sin(\omega_H j)-\sin(\omega_L j)}{\pi j}
\end{aligned}
$$

Note that you can also get b_0 from the last line by using l'Hopital's rule

Properties of the band-pass filter

$$b(L) = \sum_{j=-\infty}^{\infty} b_j L^j$$

- $b(L)$ is a polynomial of L. Consider the roots to the problem:

$$b(L) = 0$$

If $L = 1$ is a root of the problem, then we have

$$b(L) = (1 - L)\bar{b}(L) \quad \text{with } \bar{b}(1) < \infty$$

Properties of the band-pass filter

- But $L = 1$ is a root of our filter as long as $\omega_L > 0$, because then we have by construction

$$b(1) = b(e^{-i0}) = 0$$

Clearly, if you do not filter out the zero frequency then you do not induce stationarity

More on I(1) processes

- Discussion above showed

$$x_t^f = b(L)x_t \text{ is stationary even if } x_t \text{ is I(1)}$$

- This is not enough to show that the filter does what it is supposed to do, which is
  - ensure the spectrum of the filtered series is zero for the excluded frequencies
  - ensure the spectrum of the filtered series equals the spectrum of the original series for the included frequencies
- The second condition requires a definition of the spectrum for I(1) processes

Spectrum for I(1) processes

Consider an arbitrary I(1) process

$$x_t = \frac{z_t}{1-L}$$

Let

$$x_{\rho,t} = \frac{z_t}{1-\rho L}$$

For $\rho < 1$ the spectrum of $x_{\rho,t}$ is well defined

$$S_{\rho,x}(\omega) = \frac{1}{1-2\rho\cos(\omega)+\rho^2} S_z(\omega)$$

Define the spectrum of x_t as

$$S_x(\omega) = \lim_{\rho \longrightarrow 1} S_{\rho,x}(\omega)$$

This is well defined for all $\omega > 0$, but not for $\omega = 0$.

Filtered I(1) process

$$x_t^f = b(L)x_t$$

Let $b(L)$ be a band-pass filter, that is,

$$b(e^{-i\omega}) = \begin{cases} 1 & \text{if } \omega_L \leq \omega \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$$

Filtered I(1) process

- if $\omega_L > 0$, then it can be shown that
  - x_t^f is stationary (because as shown above we know that $b(1) = 0$) and
  - $S_{x^f}(\omega) = \begin{cases} S_x(\omega) & \text{if } \omega_L \leq \omega \leq \omega_H \\ 0 & \text{o.w.} \end{cases}$
- That is, using the definition of the Spectrum for I(1) processes the filter does exactly what it is supposed to do
- Proof is simple; The only tricky thing is to prove is that

$$b(e^{-i0})S_x(0) = 0$$

Practical Filter

- The filter constructed so far is two-sided and infinite order
- Implementable version would be to use

$$\tilde{b}(L) = \sum_{j=-J}^{J} b_j L^j$$

But it is not necessarily the case that

$$\tilde{b}(1) = 0$$

So instead use

$$a(L) = \sum_{j=-J}^{J} a_j L^j$$

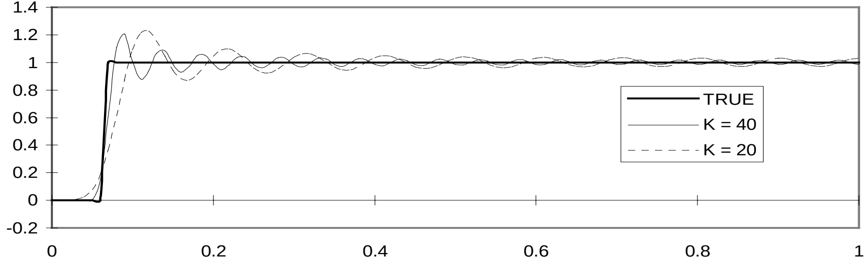
with

$$a_j = b_j + \mu \quad \text{and} \quad \mu = -\frac{\sum_{j=-J}^{J} b_j}{2J+1}$$

Properties practical filter

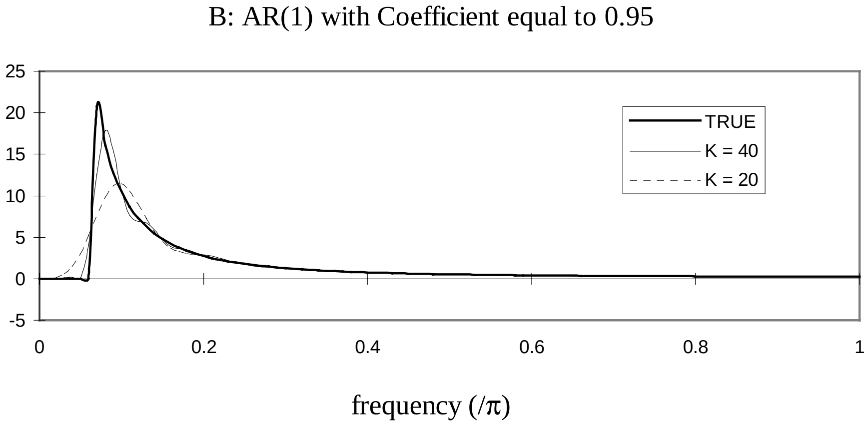
FIGURE 3.1: SPECTRA OF FILTERED PROCESSES ($\omega_1 = \pi/16, \omega_2 = \pi$)

A: White Noise (Squared Gain)

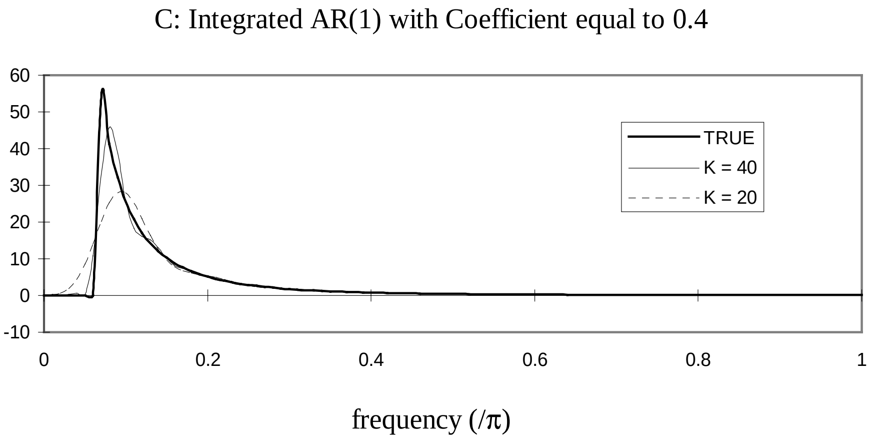


frequency (/π)

Properties practical filter



Properties practical filter



Hodrick-Prescott Filter

- The HP trend, $x_{\tau,t}$ is defined as follows

$$\{x_{\tau,t}\}_{t=1}^{T} = \arg\min_{\{x_{\tau,t}\}_{t=1}^{T}} \sum_{t=2}^{T-1} (x_t - x_{\tau,t})^2 + \lambda \sum_{t=2}^{T-1} \left\{ \left[\begin{array}{c} (x_{\tau,t+1} - x_{\tau,t}) \\ -(x_{\tau,t} - x_{\tau,t-1}) \end{array} \right]^2 \right\}$$

Hodrick-Prescott Filter

- $\lambda = 1,600$ standard for quarterly data
- the HP filter is then approximately equal to a band-pass filter with $\omega_L = \pi/16$ and $\omega_H = \pi$.
    - That is, it keeps that part of the series associated with cycles that have a period less than 32 ($=2\pi/(\pi/16)$) periods (i.e. quarters).

Understanding filtered data is tricky

- Is filtered white noise serially uncorrelated?
- Are the filtered price level and filtered output positively correlated in a model with only demand shocks?
  (example below is from Den Haan 2000)

Simple demand shock model

- Output is demand determined

$$y_t = y_t^d = D_t - P_t$$

- Demand is given by

$$\begin{aligned}(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_3 L) D_t &= \varepsilon_t \\ \text{with } -1 &< \lambda_3 < \lambda_2 < \lambda_1 \leq 1\end{aligned}$$

- Output is given by

$$y_t^s = a + bt$$

Simple demand shock model

- Equilibrium price level $\widetilde{P}_t$ satisfies

$$\widetilde{P}_t = D_t - y_t^s$$

- Actual prices adjust gradually

$$P_t = (1-\beta)\, P_{t-1} + \beta \widetilde{P}_t$$

Simple demand shock model

Solution

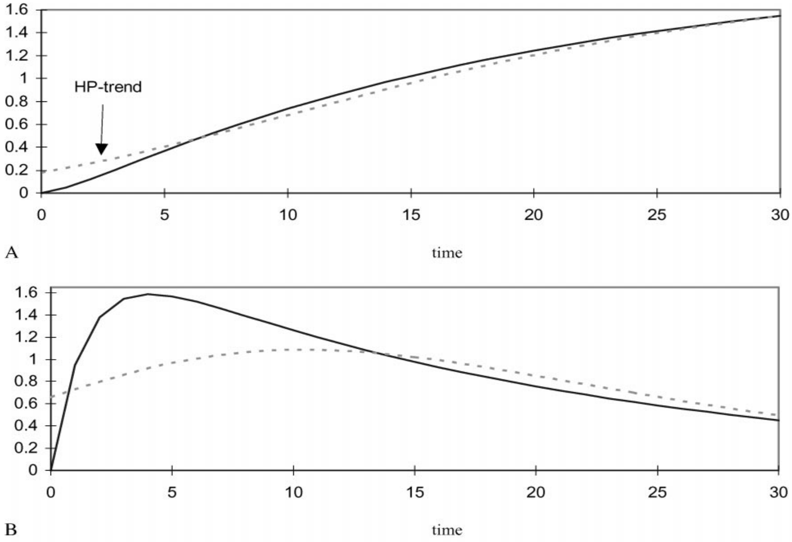
- Price level

$$P_t = \frac{\beta \varepsilon_t}{\left(1-(1-\beta)L\right)\left(1-\lambda_1 L\right)\left(1-\lambda_2 L\right)\left(1-\lambda_3 L\right)}$$

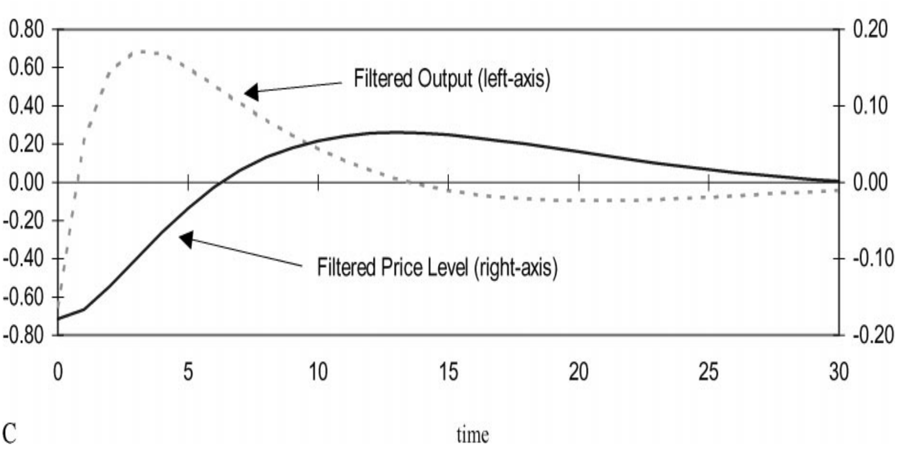
- Output

$$y_t = \frac{(1-\beta)(1-L)\,\varepsilon_t}{\left(1-(1-\beta)L\right)\left(1-\lambda_1 L\right)\left(1-\lambda_2 L\right)\left(1-\lambda_3 L\right)}$$

Positive correlation for unfiltered series



Negative correlation for filtered series



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