Chapter 1 continued

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November 20, 2010

Environment of the competitive equilibrium

- Households:
  - maximize discounted utility
  - current-period utility depends on consumption and leisure
  - own the capital stock, capital rented out each period at rate r_t
  - time endowment is equal to 1 which is split between leisure and working
  - wage rate is equal to w_t
- Firms:
  - maximize NPV of profits
  - competive input and output markets

Firm problem & FOC conditions

$$\max_{k^j, h^j} \theta \left(k_t^j\right)^{\alpha} \left(h_t^j\right)^{1-a} - r_t k_t^j - w_t h_t^j$$

$$\alpha\theta_t \left(\frac{k_t^j}{h_t^j}\right)^{\alpha-1} = r_t$$

$$(1-\alpha)\theta_t \left(\frac{k_t^j}{h_t^j}\right)^{\alpha} = w_t$$

Firm size

- CRS $\Longrightarrow$ firm size not determined
- Use representative firm without loss of generality
  - $\bar{k}_t = \sum_{j=1}^{J} k_t^j$ and $\bar{h}_t = \sum_{j=1}^{J} h_t^j$

$$\alpha\theta_t \left(\frac{\bar{k}_t}{\bar{h}_t}\right)^{\alpha-1} = r_t$$

$$(1-\alpha)\theta_t \left(\frac{\bar{k}_t}{\bar{h}_t}\right)^{\alpha} = w_t$$

- The firm thinks of these equations as follows:
  - given values for r_t and w_t choose $\bar{k}_t$ and $\bar{h}_t$.

Individual problem

$$\max_{\left\{c^i_{t+\tau},h^i_{t+\tau},k^i_{t+1+\tau}\right\}_{\tau=0}^{\infty}} \mathrm{E}\left[\sum_{\tau=0}^{\infty}\beta^{\tau}u(c^i_{t+\tau},1-h^i_{t+\tau})|I_t\right]$$

$$\text{s.t. } c^i_{t+\tau}+k^i_{t+1+\tau}\leq r_t k^i_{t+\tau}+w_t h^i_{t+\tau}+(1-\delta)k^i_{t+\tau}$$

$$k^i_{t+1+\tau}\geq 0$$

$$k_t \text{ predetermined}$$

First-order conditions

$$\frac{\partial u(c_t^i, 1-h_t^i)}{\partial c_t^i} = \lambda_t^i$$

$$\lambda_t^i w_t = -\frac{\partial u(c_t^i, 1-h_t^i)}{\partial h_t^i}$$

$$-\lambda_t^i + \beta \mathrm{E}_t \left\{ \lambda_{t+1}^i \left[r_{t+1} + 1 - \delta \right] \right\} = 0$$

$$c_t^i + k_{t+1}^i = \theta_t \left(k_t^i \right)^{\alpha} \left(h_t^i \right)^{1-\alpha} + (1-\delta) k_t^i$$

Equations

- Equations for $\bar{k}_t$ and $\bar{h}_t$ from the firm problem taking r_t and w_t as given
- Equations for k_t^i,h_t^i, c_t^i, and λ_t^i from the household problem taking r_t and w_t as given $\forall i$

- Equilibrium conditions

$$\begin{aligned} \bar{k}_t &= \sum_i k_t^i \\ \bar{h}_t &= \sum_i h_t^i \end{aligned}$$

Simplification

Suppose that all agents start with the same capital stock $\Longrightarrow$ they also make the same choices

$$k_t^i = k_t,\ h_t^i = h_t,\ c_t^i = c_t, \text{ and } \lambda_t^i = \lambda_t$$

6 equations in 6 unknowns

$$\frac{\partial u(c_t, 1-h_t)}{\partial c_t} = \lambda_t$$
$$\lambda_t w_t = -\frac{\partial u(c_t, 1-h_t)}{\partial h_t}$$
$$-\lambda_t + \beta \mathrm{E}_t \left\{ \lambda_{t+1} \left[r_{t+1} + 1 - \delta \right] \right\} = 0$$
$$c_t + k_{t+1} = \theta_t \left(k_t \right)^{\alpha} \left(h_t \right)^{1-\alpha} + (1-\delta) k_t$$
$$r_t = \alpha \theta_t \left(\frac{\bar{k}_t}{\bar{h}_t} \right)^{\alpha-1} = \alpha \theta_t \left(\frac{I k_t}{I h_t} \right)_t^{\alpha-1} = \alpha \theta_t \left(\frac{k_t}{h_t} \right)_t^{\alpha-1}$$
$$w_t = (1-\alpha) \theta_t \left(\frac{\bar{k}_t}{\bar{h}_t} \right)^{\alpha} = (1-\alpha) \theta_t \left(\frac{k_t}{h_t} \right)^{\alpha}$$

- unknowns: $c_t, k_{t+1}, h_t, \lambda_t, r_t, w_t$
- unknowns also could have been: $\bar{c}_t, \bar{k}_t, \bar{h}_t, \bar{\lambda}_t, r_t, w_t$

Definition of equilibrium

Definition (competitive equilibrium): *A competitive equilibrium consists of*

- *a consumption function,* $c(k, \bar{k}, \theta)$,
- *a labor supply function,* $h(k, \bar{k}, \theta)$,
- *a capital function,* $k_{+1}(\bar{k}, \bar{k}, \theta)$,
- *an aggregate per capita consumption function,* $\bar{c}(\bar{k}, \theta)$,
- *aggregate per capita capital function,* $\bar{k}_{+1}(\bar{k}, \theta)$,
- *aggregate per capita labor supply function,* $\bar{h}(\bar{k}, \theta)$,
- *wage function,* $w(\bar{k}, \theta)$, *and*
- *rental rate,* $r(\bar{k}, \theta)$,

Definition of equilibrium (continued)

- solve the household's optimization problem,
- solve the firm's optimization problem,
- are consistent with each other, that is,
  - $\bar{c}\left(\bar{k},\theta\right) = c(\bar{k},\bar{k},\theta)$,
  - $\bar{k}_{+1}\left(\bar{k},\theta\right) = k_{+1}(\bar{k},\bar{k},\theta)$, and
  - $\bar{h}\left(\bar{k},\theta\right) = h(\bar{k},\bar{k},\theta)\,\forall\bar{k},\forall\theta$
- satisfy the aggregate budget constraint:
  - $\bar{c}(\bar{k},\theta) + \bar{k}_{+1}(\bar{k},\theta) = \theta\bar{k}^{\alpha}\bar{h}^{1-\alpha} + (1-\delta)\bar{k}$.

Representative agent and aggregate

- the equation above distinguish between the choices and state variables of the individual, c, k_{+1}, and h, and the aggregate variables, $\bar{c}$, $\bar{k}_{+1}$, and $\bar{h}$.
- In a representative agent framework these are the same and in practice we only solve for $\bar{c}(\bar{k})$, $\bar{k}_{+1}(\bar{k})$, and $\bar{h}(\bar{k})$.
- But, it is important to understand that the complete solution allows us to answer the question what will happen with (say) an individual agent's consumption when his individual capital stock increases
  - Note that $\bar{k}$ does not change when the capital stock of one individual increases (because the effect of each individual on the economy is nil)
- Instead of using aggregate variables you can also express them as per capita variables

Social Planner & CE

What is a social planner?

- Maximizes utility but only worries about feasibility and not prices and transfers
- That is, the social planner problem for this CE is the same as our Robinson Crusoe economy
- If you substitute out the rental rate and the wage rate, then you see that the equations of the competitive equilibrium are the same as those of the social planner.

Many different agents

Environment

- Endowment economy
- Agents are ex ante the same but receive different realizations of the endowment
- Complete asset markets

Result:

- Economy can be represented with a representative agent economy

Optimization problem

$$\max_{c^i, b^{1,i}_{+1}, \cdots, b^{J,i}_{+1}} \frac{\left(c^i\right)^{1-\gamma}}{1-\gamma} + \beta E v(b^{1,i}_{+1}, \cdots, b^{J,i}_{+1})$$

$$s.t. c^i + \sum_{j=1}^{J} q^j b^{j,i}_{+1} = y^i + \sum_{j=1}^{J} I(j^*) b^{j,i}$$

$$b^{j,i}_{+1} > \overline{b} < 0$$

First-order conditions

$$q^j \left(c^i\right)^{-\gamma} = \beta \left(c_{+1}^{j,i}\right)^{-\gamma} \text{prob}(j) \; \forall j$$

Explicit aggregation

$$
\begin{aligned}
c^i &= \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} c^{j,i}_{+1} \ \forall j \ \text{ or} \\
C &= \left(\frac{\beta \text{prob}(j)}{q^j}\right)^{-1/\gamma} C^{j}_{+1} \ \forall j,
\end{aligned}
$$

$$
q^j \left(C\right)^{-\gamma} = \beta \left(C^j_{+1}\right)^{-\gamma} \ \forall j
$$

$$
q^j \left(Y\right)^{-\gamma} = \beta \left(Y^j_{+1}\right)^{-\gamma} \ \forall j
$$

Representative agent economy

$$\max_{C,B^1_{+1},\cdots,B^J_{+1}} \frac{(C)^{1-\gamma}}{1-\gamma} + \beta \mathrm{E} v(B^1_{+1},\cdots,B^J_{+1})$$

$$s.t. C^i + \sum_{j=1}^{J} q^j B^j_{+1} = Y + \sum_{j=1}^{J} I(j^*) B^j$$

$$B^j_{+1} > \overline{b} < 0$$

Any difference?