# Solving Models with Heterogeneous Agents : Limited History Dependence

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# Perturbation and employment shocks

- Suppose agents are subject to idiosyncratic unemployment shocks
    - $\varepsilon_{i,t} \in \{0,1\}$ or $\varepsilon_{i,t} \in \{u,e\}$
- Could you solve such models using perturbation methods?

# Perturbation and employment shocks

- To simplify discussion: no aggregate shocks
- FOCs:

for employed
$$
\begin{aligned}
c_{i,t} + k_{i,t} &= (1 + r - \delta)k_{i,t-1} + w \\
c_{i,t}^{-\gamma} &= \beta \mathbb{E}_t \left[ c_{i,t-1}^{-\gamma}(1 + r - \delta) \right]
\end{aligned}
$$

for unemployed
$$
\begin{aligned}
c_{i,t} + k_{i,t} &= (1 + r - \delta)k_{i,t-1} + b \\
c_{i,t}^{-\gamma} &= \beta \mathbb{E}_t \left[ c_{i,t-1}^{-\gamma}(1 + r - \delta) \right]
\end{aligned}
$$

# Perturbation and employment shocks

Why couldn't we simply give the following model to Dynare?

for employed

$$c_{e,t} + k_{e,t} = (1 + r - \delta)k_{e,t-1} + w$$
$$c_{e,t}^{-\gamma} = \beta \mathbb{E}_t \left[ c_{t-1}^{-\gamma}(1 + r - \delta) \right]$$

for unemployed

$$c_{u,t} + k_{i,t} = (1 + r - \delta)k_{u,t-1} + b$$
$$c_{u,t}^{-\gamma} = \beta \mathbb{E}_t \left[ c_{t-1}^{-\gamma}(1 + r - \delta) \right]$$

variables

$$c_{e,t}, c_{u,t}, k_{e,t}, k_{u,t}$$

1. Typically we use borrowing constraints to keep problem well defined, but we could use smooth penalty functions instead.
2. What is the more fundamental problem?

# Perturbation and employment shocks

- Koen Vermeylen of the University of Amsterdam was (I think) the first to realize this could be done. Vermeylen (2006) uses system of previous slide:
  1. keep track of *both* $k_{e,t}$ and $k_{u,t}$ *for all t*
  2. uses a well-chosen AR(1) process, $z_t$, that
     1. in simulation, shocks are such that $z_t \in \{0,1\}$,
     2. selects current employement status, *and*
     3. the actual currentcapital stock: $k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$.
  3. Substitute out $k_{t-1}$
- See appendix for details

# LeGrand-Ragot (LGR) environment

- Exogenous aggregate risk affects rental rate of capital and wage rate
- Exogenous aggregate risk does *not* affect employment risk
  - but this can be done (as shown at end of slides)
- Incomplete markets
  - short-sell constraint and saving only through capital
  - *some* joint risk sharing as discussed below
- Preference shocks to get realistic wealth distribution
- An unemployed worker works $\delta$ hours at home to produce $\delta$ goods (parameters are chosen such that agents do not prefer to work less than $\delta$)

# Key approximating assumption

Key approximation step: All agents with the same employment history for the last $N$ periods are identical

- If $N = 2$, then there are 4 types:
  *uu, ue, eu, ee*
- If $N = 3$, then there are 8 types:
  *uuu,uue,ueu,uee,euu,eue,eeu,eee*
  - (in general, if there are $E$ individual states then there are $(E + 1)^N$ types; here $E = 2$)
- Original model: $N = \infty$, that is, an infinite number of different agents

# Stories representing approximation

- LGR propose two "stories/models" so that the set of equations given to the computer looks like *an actual economy* and not just an approximation to the original model
    1. quasi-planner
    2. decentralized version with particular insurance mechanism
- This is useful, for example, to understand whether the set of equations of the approximation is well behaved

# Quasi planner "story"

- Agents with the same employment history for the last $N$ periods have the same consumption and make the same savings choice *independent of the wealth they bring into period t*
- This savings choice is made by the quasi-planner
- The quasi-planner does take prices as given (in contrast to the conventional social planner)

# Quasi-planner "story"

- Beginning of period $t$ :
  - all agents with the same $N$-period employment history go to the same "island"
  - their savings are pooled
  - quasi-planner chooses consumption and savings
- End of period $t$ : All agents are entitled to an equal share of the savings
  - Thus, quasi-planner cannot condition on *next-period's* unemployment status. This mimics market incompleteness

# Quasi-planner model

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t \sum_{e^N\in\varepsilon^N} S_{t,e^N}\xi_{e^N}U\left(c_{t,e^N},l_{t,e^N}\right)\right]$$

s.t.

$$\begin{aligned}
a_{t,e^N}+c_{t,e^N} &= w_t l_{t,e^N} n_{t,e^N} + \delta 1_{e^N=0} + (1+r_t)\,\widetilde{a}_{t,e^N} \quad \forall e^N \in \varepsilon^N \\
a_{t,e^N} &\geq 0 \quad \forall e^N \in \varepsilon^N \\
\widetilde{a}_{t,e^N} &= \sum_{\widetilde{e}^N\in\varepsilon^N} \frac{S_{t-1,\widetilde{e}^N}}{S_{t,e^N}} \Pi_{t-1,(\widetilde{e}^N,e^N)} a_{t-1,\widetilde{e}^N} \quad \forall e^N \in \varepsilon^N \\
S_{t+1,e^N} &= \Pi_t S_{t,e^N} \\
l_{t,e^N} &\geq 0
\end{aligned}$$

# Quasi-planner model

- Index to indicate a particular type: $e^N \in \varepsilon^N$
- $S_{t,e^N}$ : population size island $e^N$
- $\widetilde{a}_{t,e^N}$ : per capita beginning-of-period wealth on island $e^N$
    - $S_{t,e^N}\widetilde{a}_{t,e^N}$ equals sum of savings brought to island $e^N$ from different islands
- $1_{e^N=0}$ : indicator function if agents on this island are unemployed
- $n_{t,e^N}$ : idiosyncratic productivity agents on island $e^N$ $n_{t,e^N} = 0$ if $1_{e^N=0} = 1)$
- $\xi_{e^N}$ : preference parameter
    - agents with different employment histories have a different utility function
- $\Pi_t$ : transition matrix for the full $N$-period employment state
    - examples below

# Quasi-planner FOCs

$$\xi_{e^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) + \nu_{t,e^N}$$
$$=$$
$$\beta \mathbb{E}_t\left[\sum_{\widehat{e}^N \in \varepsilon^N} \Pi_{t,(e^N,\widehat{e}^N)} \xi_{\widehat{e}^N} U_c\left(c_{t+1,e^N}, l_{t+1,e^N}\right)\left(1 + r_{t+1}\right)\right]$$
$$\nu_{t,e^N} a_{t,e^N} = 0,\ a_{t,e^N} \geq 0,\ \nu_{t,e^N} \geq 0$$
$$w_t n_{e_t^N} U_c\left(c_{t,e^N}, l_{t,e^N}\right) = -U_l\left(c_{t,e^N}, l_{t,e^N}\right) \text{ if } n_{t,e^N} > 0$$
$$l_{t,e^N} = \delta \text{ if } n_{t,e^N} = 0$$

# Quasi-planner FOCs

- Note that the population sizes drop out!
  - going to a large $S_{t,e^N}$ island is *bad* because you have to share your wealth with more agents
  - going to a large $S_{t,e^N}$ island is *good* because the social planner gives it a larger weight
  - these effecs exactly offset each other
- Note that the *linearized* Euler equation captures precautionary savings

# Other model equations

aggregate labor supply
$$L_t = \sum_{e^N \in \varepsilon^N} S_{t,e^N} n_{t,e^N} l_{t,e^N}$$

aggregate savings
$$K_t = \sum_{e^N \in \varepsilon^N} S_{t,e^N} a_{t,e^N} = \sum_{e^N \in \varepsilon^N} S_{t+1,e^N} \widetilde{a}_{t+1,e^N}$$

wage rate
$$w_t = (1-\alpha) A_{t-1} \left(\frac{K_{t-1}}{L_t}\right)^{\alpha}$$

rental rate
$$r_t = \alpha A_{t-1} \left(\frac{K_{t-1}}{L_t}\right)^{\alpha-1} - depreciation$$

productivity
$$A_t = 1 + u_t$$
$$u_t = \rho u_{t-1} + e_t$$

# Specific assumptions

- Greenwood, Hercowitz, Huffman preferences

$$U_c\left(c_{t+1,e^N}, l_{t+1,e^N}\right) = \frac{\left(c_{t+1,e^N} - \frac{l_{t+1,e^N}^{1+1/\phi}}{1+1/\phi}\right)^{1-\gamma} - 1}{1-\gamma}$$

- $\Longrightarrow$ first-order condition for employed becomes

$$w_t n_{e_t^N} = l_{t,e^N}^{1/\phi}$$

# Specific assumptions

- If $n_{e_t^N}$ (just as aggregate productivity) is known in period $t$, then $L_t$ is known in period $t \implies r_t$ is known in period $t$ (risk-free $r_t$ means capital would be perfect substitute to risk-free government bonds)
- In fact, it is assume that $n_{e_t^N} = 1$ for all employed agents
- $\Pi_t$ is constant $\implies$ unemployment rate is constant
- $N = 4$

# Constructing transition matrix

16 groups:

| unemployed | employed |
| --- | --- |
| 1. uuuu | 9. euuu |
| 2. uuue | 10. euue |
| 3. uueu | 11. eueu |
| 4. uuee | 12. euee |
| 5. ueuu | 13. eeuu |
| 6. ueue | 14. eeue |
| 7. ueeu | 15.eeeu |
| 8. ueee | 16.eeee |

- probability to become employed for unemployed equals 0.5
- probability to become unemployed for employed equals 0.2

$$\Pi = \begin{bmatrix}
.5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 \\
.5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .5 & .5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8 & .8
\end{bmatrix}$$

# The tricky bit

- You have to figure out by trial and error (and some economic thinking) which group will be at the constraint
- Things would be problematic if that depends on the aggregate state
  - (less likely to be problematic if aggregate fluctuations are small)
- Here, only group 1 turns out to be at the constraint

# Some Dynare equations

- Budget constraint for group 1, uuuu, thus currently unemployed

  c1=delta+(1+r)*0.5*(S2*a2(-1)+S1*a1(-1))/S1-a1

  - this group gets members from groups 1 & 2

- First-order condition for group 1

  a1 = 0;

# Some Dynare equations

- Budget constraint for group 2, uuue, thus currently unemployed

  c2=delta+(1+r)*0.5*(S4*a4(-1)+S3*a3(-1))/S2-a2

  - this group gets members from groups 3 & 4

# Some Dynare equations

- First-order condition for group 2

```
weight2*(c2-delta^(1+1/phi)/(1+1/phi))^-sigma
=
beta*(1+r(+1))*
(
0.5*weight9*(c9(+1)-le(+1)^(1+1/phi)/(1+1/phi))^-sigma
+
0.5*weight1*(c1(+1)-delta^(1+1/phi)/(1+1/phi))^-sigma
);
```

  - Members of this group can go to group 1, *uuuu*, or group 9, *euuu*, with equal probability

# Some Dynare equations

- Budget constraint for group 9, euuu, thus currently employed

  c9=w*le+(1+r)*0.5*(S2*a2(-1)+S1*a1(-1))/S9-a9

  - this group gets members from groups 1 & 2

# Some Dynare equations

- First-order condition for group 9

```
weight9*(c9-le^(1+1/phi)/(1+1/phi))^-sigma
=
beta*(1+r(+1))*
(
0.8*weight13*(c13(+1)-le(+1)^(1+1/phi)/(1+1/phi))^-sigma
+
0.2*weight5*(c5(+1)-delta^(1+1/phi)/(1+1/phi))^-sigma
);
```

  - Members of this group can go to group 5, *ueuu*, or group 13, *eeuu*, with 0.2 and 0.8 probability, respectively

# Modification: State dependent unemployment

State dependent $\Pi$

If $N = 2$, then one could have

$$S_t = \begin{bmatrix} .5 - \eta_{u\ u} A_{t-1} & .5 - \eta_{ue} A_{t-1} & 0 & 0 \\ 0 & 0 & .2 - \eta_{e\ u} A_{t-1} & .2 - \eta_{e\ e} A_{t-1} \\ .5 + \eta_{u\ u} A_{t-1} & .5 + \eta_{ue} A_{t-1} & 0 & 0 \\ 0 & 0 & .8 + \eta_{e\ u} A_{t-1} & .8 + \eta_{e\ e} A_{t-1} \end{bmatrix} S_{t-1}$$

# Modification: State dependent unemployment

- Note that the columns sum up to 1
- Following LGR, dependence is on $A_{t-1}$, but could also be $A_{t-1}$
- !!! This works without complications *only if* the aggregate state still does not matter for which group is at the borrowing constraint

# Other modifications

- Productivity of the employed could be different
    - for example, those who were recently unemployed are less productive

# Appendix: Vermeylen approach

- Consider the following model

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_t = \exp(\theta_t) k_{t-1}^{\alpha} + (1-\delta) k_{t-1}$$

$$\theta_{t+1} = \begin{array}{c} \theta_L \text{ with probability } p(\theta|\theta_t) \\ \theta_H \text{ with probability } 1 - p(\theta_t) \end{array}$$

- First-order perturbation:

$$k_t = \bar{k} + h_k(k_{t-1} - \bar{k}) + h_\theta(\theta_t - \bar{\theta})$$

- Thus, $h_k$ is the same independent of the value of $\theta$

# First-order conditions

- policy function when $\theta_t = \theta_L$: $k_L(k_{t-1})$
- policy function when $\theta_t = \theta_H$: $k_H(k_{t-1})$
- Euler equation when $\theta_t = \theta_L$

$$(\theta_L k_{t-1}^{\alpha} - k_{L,t})^{-\gamma} = \begin{array}{c} p_{LL}\beta(\theta_L k_{L,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1} + 1 - \delta) \\ (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1} + 1 - \delta \end{array}$$

- Euler equation when $\theta_t = \theta_H$

$$(\theta_H k_{t-1}^{\alpha} - k_{H,t})^{-\gamma} = \begin{array}{c} (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha} - k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1} + 1 - \\ p_{HH}\beta(\theta_H k_{H,t}^{\alpha} - k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1} + 1 - \delta) \end{array}$$

- Auxiliary equation

$$k_{t-1} = (1 - z_{t-1})k_{L,t-1} + z_{t-1}k_{H,t-1}$$

- Now, $\theta_L$ & $\theta_H$ are fixed parameters and $z_t$ is the stochastic variable.

# New system with new variables

- Substitute out $k_{t-1}$. Now $z_t$ enters the orginal Euler equations
- $k_{L,t}$ and $k_{H,t}$ have different steady state values
- Let the law of motion for $z_t$ be given by

$$z_t - \bar{z} = \rho(z_{t-1})\left(z_{t-1} - \bar{z}\right) + \varepsilon_t. \tag{1}$$

$$
\begin{aligned}
&\left(\theta_L\left((1-z_{t-1})k_{L,t-1}+z_{t-1}k_{H,t-1}\right)^{\alpha}-k_{L,t}\right)^{-\gamma}\\
= \; & p_{LL}\beta(\theta_L k_{L,t}^{\alpha}-k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{L,t}^{\alpha-1}+1-\delta)\\
& (1-p_{LL})\beta(\theta_H k_{L,t}^{\alpha}-k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{L,t}^{\alpha-1}+1-\delta)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\theta_H\left((1-z_{t-1})k_{L,t-1}+z_{t-1}k_{H,t-1}\right)^{\alpha}-k_{H,t}\right)^{-\gamma}\\
= \; & (1-p_{HH})\beta(\theta_L k_{H,t}^{\alpha}-k_{L,t+1})^{-\gamma}(\alpha\theta_L k_{H,t}^{\alpha-1}+1-\delta)\\
& p_{HH}\beta(\theta_H k_{H,t}^{\alpha}-k_{H,t+1})^{-\gamma}(\alpha\theta_H k_{H,t}^{\alpha-1}+1-\delta)
\end{aligned}
$$

$$
z_t-\bar{z}=\rho(z_{t-1})\left(z_{t-1}-\bar{z}\right)+\varepsilon_t
$$

- You could also use the linearized version of 1, since that is what will be using anyway

$$z_t = \bar{z} + \rho(\bar{z})(z_{t-1} - \bar{z}) + \varepsilon_t. \tag{2}$$

- The unconditional mean for $z_t$, $\bar{z} = \mathsf{E}[z_t]$, equals

$$\bar{z} = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}\theta_H + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}\theta_L$$

- The unconditional mean of $\rho(z_{t-1})$, $\rho(\bar{z}) =\mathsf{E}[\rho(z_t)]$, equals

$$\rho(\bar{z}) = \frac{1 - p_{LL}}{2 - p_{LL} - p_{HH}}(2p_{HH} - 1) + \frac{1 - p_{HH}}{2 - p_{HH} - p_{LL}}(2p_{LL} - 1)$$

# Are new specification and original model consistent?

- In simulation use 1 not 2; so you have to do your own simulation
- We need
  - $z_t \in \{0,1\}$
  - $\mathsf{E}[\varepsilon_t|z_{t-1}=0] = E\left[\varepsilon_t|z_{t-1}=1\right] = 0$
  - Conditional autocorrelations have to be correct
    - $\rho(1) = 2p_{HH} - 1$
    - $\rho(0) = 2p_{LL} - 1$

# Are new specification and original model consistent?

To get that

$$z_t = z_{t-1} \text{ with prob } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

set

$$\varepsilon_t = (1 - \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob } z_{t-1}p_{HH} + (1 - z_{t-1})p_{LL}$$

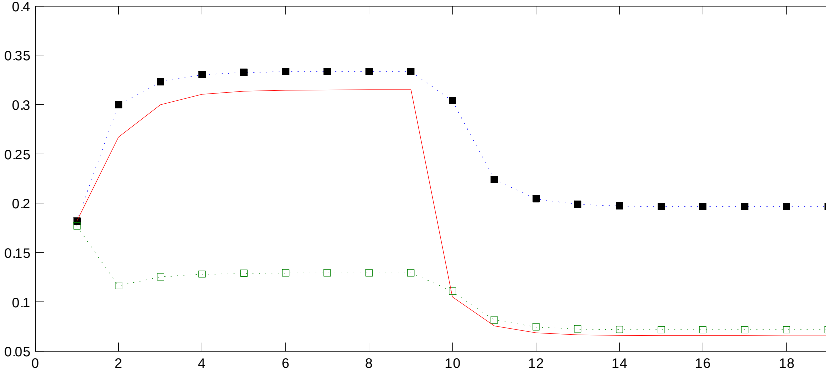
To get that

$$z_t = 1 - z_{t-1} \text{ with prob } z_{t-1}(1 - p_{HL}) + (1 - z_{t-1})(1 - p_{LL})$$

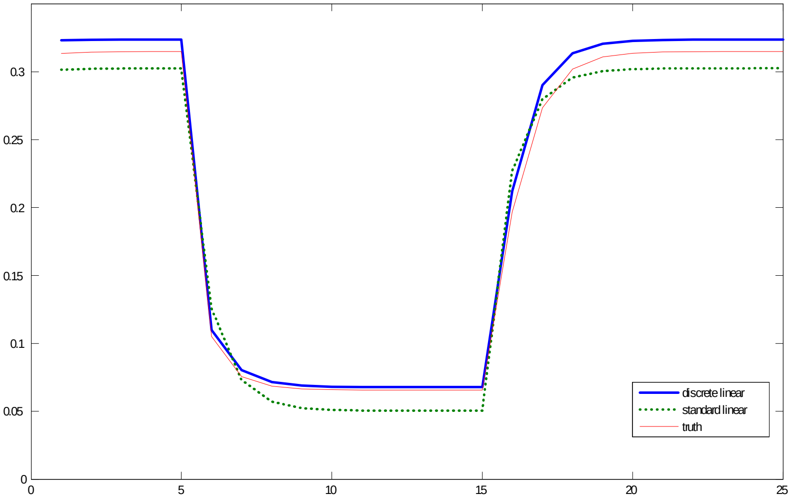
set

$$\varepsilon_t = -(1 + \rho(z_{t-1}))(z_{t-1} - \bar{z}) \text{ with prob } z_{t-1}(1 - p_{HH}) + (1 - z_{t-1})(1$$

# log-linear discrete linearization



# linear discrete versus standard linearization



# References


- Challe, Eduard, and Xavier Ragot, 2015, Precautionary savings over the Business Cycle, Economic Journal 126, 135-164.
- Le Grand, Francois, and Xavier Ragot, 2016, Heterogeneous Agents: A reduced Heterogeneity Framework, manuscript.
- Vermeylen, Koen, 2006, Heterogeneous Agents and Uninsurable Idiosyncratic Employment Shocks in a Linearlized Dynamic General Equilibrium Model, Journal of Money, Credit and Banking 38, 837-846.