Chapter 2

Equilibrium Models

2.1 Introduction

The purpose of this chapter is twofold. First, we will introduce the reader to some popular dynamic equilibrium models used in the literature. The second goal of this chapter is to improve the reader's skill in working with equilibrium models. In particular, we construct systems of n equations in n unknowns to characterize the solution of the model, determine the set of state variables, calculate steady states, analyze properties of the model without explicitly solving it, and compare the behavior of economic variables in the competitive equilibrium to the behavior of these variables if they are chosen by a social planner. In Section 2.2, we consider an extension of the model developed in Chapter 1 in which the government issues fiat money and money, besides a source of wealth also fascilitates transactions. In Section 2.3 we consider non-monetary and monetary overlapping generations models. In these models, agents only live for a finite time period and at each point in time cohorts of different ages are alive.

2.2 Monetary models with infinitely-lived agents

2.2.1 Specification of the model

Households in this economy solve the following optimization problem:

$$\begin{array}{c}\max\limits_{\left\{\begin{array}{c}c_{t+j},h_{t+j},k_{t+j+1},\\ v_{t+j},M_{t+j+1},B_{t+j+1}\end{array}\right\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty}\beta^j u(c_{t+j},1-h_{t+j}-v_{t+j})|\, I_t\right]\\ \text{s.t.}\quad c_{t+j}+k_{t+1+j}+\frac{M_{t+1+j}}{p_{t+j}}+q_{t+j}\frac{B_{t+1+j}}{p_{t+j}}+\tau_{t+j}=\\ \theta_{t+j}f(k_{t+j},h_{t+j})+(1-\delta)k_{t+j}+\frac{M_{t+j}}{p_{t+j}}+\frac{B_{t+j}}{p_{t+j}}\\ v_{t+j}=v\left(c_{t+j},\frac{M_{t+j}}{p_{t+j}}\right)\\ k_t, M_t, \text{ and } B_t \text{ predetermined}\end{array} \tag{2.1}$$

Here c_t stands for consumption, h_t for labor supply, v_t for shopping time, k_t for beginning-of-period t capital, M_t for beginning-of-period t nominal money balances, p_t for the price level, τ_t for lump-sum taxes, θ_t for the productivity shock, and B_t for the number of bonds bought at period $t-1$. Also q_{t-1} is the price of a bond bought in period $t-1$ that delivers one unit of money in period t. Leisure in this economy is equal to $1-h_t-v_t$. The amount of time spent shopping is a function of c_t and real money balances $m_t = M_t/p_t$ with

$$\frac{\partial v(c,m)}{\partial c} > 0 \text{ and } \frac{\partial v(c,m)}{\partial m} < 0. \tag{2.2}$$

Thus, the higher the amount of consumption the higher the amount of time spent shopping and the higher the amount of real money balances the smaller the amount of time spent shopping. At the end of this section, we will give a more detailed motivation for the shopping time function.

The budget constraint for the government is given by

$$\frac{M^s_{t+1}-M^s_t}{p_t}+\frac{q_tB^s_{t+1}-B^s_t}{p_t}+\tau_t=g_t, \tag{2.3}$$

where g_t is the per capita amount of government expenditures, M^s_t is the (per capita) money supply, and B^s_t is the (per capita) bond supply. According to this budget constraint, the government can finance government expenditures through seigniorage, by issuing bonds, and by levying taxes. The constraint implies that if the government chooses three of their four instruments, then the fourth one is pinned down. We will assume that g_t, M^s_t, and B^s_t are exogenous processes and that τ_t is solved from 2.3. In particular, suppose that

$$\ln(g_t) = \gamma_0+\gamma_1\ln(g_{t-1})+\varepsilon^g_t, \tag{2.4}$$

$$\ln(B^s_{t+1}) = \phi_0+\phi_1\ln(B^s_t)+\varepsilon^B_t, \text{ and} \tag{2.5}$$

$$\ln(M^s_{t+1}/M^s_t) = \mu_0+u_1\ln(M^s_t/M^s_{t-1})+\varepsilon^M_t, \tag{2.6}$$

where $\varepsilon^g_t, \varepsilon^B_t$, and ε^M_t are independent white-noise error terms.

Deriving the shopping-time function

The idea is that to acquire a consumption level equal to c_t requires producing *acquisition services* $a_t \geq c_t$. These acquisition services can be produced using real money balances and shopping time as inputs, just as capital and labor are used as inputs in the production function. Real money balances reduce the amount of resources needed to acquire a certain amount of consumption, for example because a higher level of real money balances means that less time has to be spent searching for commodities that can be bought on credit and bargaining about the interest payments. The assumption that one needs a costly resource like time to acquire consumption seems like a weak assumption even though it implicitly argues that those in this world who enjoy shopping are a minority. Don't forget, however, that this is supposed to be a macro model. So part of the costs of "shopping" are the costs of the banking sector to check for credit ratings, etc..

Suppose that the function that specifies how the inputs money and shopping time can be used to produce acquisition services is equal to

$$a_t = \bar{\xi} m_t^{\kappa} v_t^{1-\kappa}. \tag{2.7}$$

If κ is equal to one then shopping time is not productive and only real money balances are needed to acquire consumption. Moreover if $\bar{\xi}$ is equal to one as well, then we would have the standard cash-in-advance specification. That is,

$$c_t \leq a_t = m_t. \tag{2.8}$$

For $0 < \kappa < 1$ we have

$$c_t = a_t = \bar{\xi} m_t^{\kappa} v_t^{1-\kappa}. \tag{2.9}$$

Note that we have imposed the equality that $c_t = a_t$ in 2.9 because for regular utility functions agents would never use more shopping time than is absolutely needed.[1] Rewriting Equation 2.9 gives

$$v_t = \xi c_t^{1/(1-\kappa)} m_t^{-\kappa/(1-\kappa)}, \tag{2.10}$$

where $\xi = (1/\bar{\xi})^{1/(1-\kappa)}$. If we substitute the shopping-time function into the current-period utility function then we get a utility function that depends on consumption, labor supply, and real money balances. The shopping-time model is, thus, a special case of money-in-the-utility (MIU) models in which just owning real money balances provides utility.[2]

[1] We did not impose this restriction in Equation 2.8 because money balances M_t are chosen in period $t-1$. The agent would not like to use any more money than is needed since money doesn't earn any interest and bonds do. However, he cannot predict perfectly how much money is going to be needed. In particular, c_t and p_t are not known in period $t-1$ and it is, therefore, not necessarily true that 2.8 holds with equality.

[2] Make sure you don't confuse this reason for why real money balances have utility with the reason that (real) money balances provide utility indirectly because they represent a source of wealth.

2.2.2 First-order conditions and definition of equilibrium

The first-order conditions for the agent's problem are the following:

$$\lambda_t = \frac{\partial u(c_t, l_t)}{\partial c_t} - \frac{\partial u(c_t, l_t)}{\partial l_t}\frac{\partial v(c_t, m_t)}{\partial c_t}, \tag{2.11a}$$

$$\lambda_t = \beta \mathrm{E}\left[\lambda_{t+1}\left(\theta_{t+1}\frac{\partial f(k_{t+1}, h_{t+1})}{\partial k_{t+1}} + 1 - \delta\right) \mid I_t\right], \tag{2.11b}$$

$$\frac{\partial u(c_t, l_t)}{\partial l_t} = \lambda_t \theta_t \frac{\partial f(k_t, h_t)}{\partial h_t}, \tag{2.11c}$$

$$\frac{q_t \lambda_t}{p_t} = \beta \mathrm{E}_t\left[\frac{\lambda_{t+1}}{p_{t+1}}\right], \tag{2.11d}$$

$$\frac{\lambda_t}{p_t} = \beta \mathrm{E}\left[\frac{\lambda_{t+1} - \frac{\partial u(c_{t+1}, l_{t+1})}{\partial l_{t+1}}\frac{\partial v(c_{t+1}, m_{t+1})}{\partial m_{t+1}}}{p_{t+1}} \mid I_t\right], \tag{2.11e}$$

$$c_t + k_{t+1} + \frac{M_{t+1}}{p_t} + q_t \frac{B_{t+1}}{p_t} + \tau_t = \tag{2.11f}$$
$$\theta_t f(k_t, h_t) + (1-\delta)k_t + \frac{M_t}{p_t} + \frac{B_t}{p_t},$$

$$\lim_{J \to \infty} \mathrm{E}\left[\beta^{J-1}\lambda_J k_{J+1} | I_t\right] = 0, \tag{2.11g}$$

$$\lim_{J \to \infty} \mathrm{E}\left[\beta^{J-1} q_J \frac{\lambda_J}{p_J} B_{J+1} | I_t\right], \text{ and} \tag{2.11h}$$

$$\lim_{J \to \infty} \mathrm{E}\left[\beta^{J-1}\frac{\lambda_J}{p_J} M_{J+1} | I_t\right], \tag{2.11i}$$

where $l_t = 1 - h_t - v(c_t, m_t)$. It would be a good exercise to derive these first-order conditions using the Lagrangian for the sequence problem.

Suppose that each agent in the economy has the same starting values, thus, $M_t = M_t^s, B_t = B_t^s$, and $k_t = K_t$, where K_t is the per capita capital stock. Since all agents are the same, the agents' demand functions are the same and the economy is in equilibrium when the quantities demanded by our representative agent are equal to the per capita supplied quantities. Thus,

$$M_{t+1} = M_{t+1}^s \tag{2.12a}$$
$$B_{t+1} = B_{t+1}^s \tag{2.12b}$$

A competitive equilibrium consist of solutions for c_t, h_t, k_{t+1}, M_{t+1}, B_{t+1}, λ_t, p_t, and q_t that satisfy the equations in 2.11 and 2.12. Since $M_{t+1} = M_{t+1}^s$ and $B_{t+1} = B_{t+1}^s$ we can also define a competitive equilibrium as a set of solutions for $c_t, h_t, k_{t+1}, \lambda_t, p_t$, and q_t that satisfy the equations in 2.11. In that case M_{t+1} and B_{t+1} are exogenous variables. Working with a smaller set of endogenous variables is often convenient, if you try to numerically solve the model. But you have to realize that the individual doesn't act as if he has to set M_{t+1} equal to M^s_{t+1}. He thinks he is free to choose any M_{t+1}. At equilibrium prices, however, it is optimal to choose a value for M_{t+1} that is equal to M^s_{t+1}.

State variables

A solution to this model would consist of a consumption function $c(s_t)$, a capital function $k(s_t)$, a labor supply function $h(s_t)$, a money demand function $M(s_t)$, a bond demand function $B(s_t)$, a price function $p(s^a_t)$, a bond price function $q(s^a_t)$, and a tax function $\tau(s^a_t)$, where s_t is a vector of state variables relevant for the individual and s^a_t is a vector of aggregate state variables. Let's think about what the state variables in this problem are. Clearly relevant for the agent's choices are the capital stock, k_t, his money holdings, M_t, his bond holdings, B_t, and the productivity shock. In addition, he cares about current and future values of the tax level, the bond price, and the price level. Current values of these three variables are known but are bad candidates to serve as state variables since they are not predetermined. Moreover, since we typically don't know whether these variables are Markov processes or if we do know of what order, we wouldn't know how many lags to include. But we can come up with a list of variables that will determine current and future values of the tax level, the bond price, and the price level. Those are M^s_{t+1}/M^s_t, M^s_t, B^s_{t+1}, B^s_t, K_t, and g_t. Note that M^s_t, B^s_t, and K_t are included because they represent wealth components of the average agent in this economy. The growth rate of money, M^s_{t+1}/M^s_t, is included because it determines together with M^s_t the money supply in period t and because it is a sufficient predictor for future money growth rates. For similar reasons B^s_{t+1} and g_t are included because they, among other things, affect tax rates. This gives $s_t = [M_t,\ B_t,\ k_t,\ M^s_{t+1}/M^s_t,\ M^s_t,\ B^s_{t+1},\ B^s_t,\ K_t,\ g_t]$ and $s^a_t = [M^s_{t+1}/M^s_t,\ M^s_t,\ B^s_{t+1}, B^s_t,\ K_t,\ g_t]$. Since all agents are identical and have been identical in the past, it will always be the case that $k_t = K_t$, $M_t = M^s_t$, and $B_t = B^s_t$. When you use this condition then s_t would be equal to $[M_t,\ B_t,\ k_t,\ M_{t+1}/M_t,\ B_{t+1},\ g_t]$ and s^a_t would be equal to $[M_{t+1}/M_t,\ M_t,\ B_{t+1}, B_t,\ K_t,\ g_t]$. But in principle, our models allows us to ask and answer the question how our agent (who is only a really small part of this economy) would behave if his own capital stock is say 5% higher than the average capital stock. When we reduce the set of state variables we cannot do this anymore [3]

2.2.3 Analyzing the competitive equilibrium without explicitly solving it

Even without explicitly solving for policy functions and equilibrium prices one can sometimes determine important properties of the equilibrium solutions. In this section we discuss two such properties. The first one is Ricardian equivalence and the second is money neutrality. Later in this chapter we will provide two more examples. In Section 2.2.5 we will analyze optimality properties of

[3] Now that we have defined the state variables it would be a good exercise to derive the first-order conditions in 2.11 again using the Bellman equation.

the competitive equilibrium and in Section 2.2.6 we will determine whether a cash-in-advance constraint is binding.

Ricardian Equivalence

A model is said to satisfy Ricardian Equivalence if a change in the time-path of government debt, keeping government spending fixed, does not affect equilibrium prices or the individual's choices for consumption, capital, labor supply, and real money balances. The amount of taxes is of course affected by a change in B_{t+1} and an increase in B_{t+1} would decrease government savings. Under Richardian Equivalence, however, this decrease is exactly offset by an increase in private savings and aggregate savings remains the same. If a model satisfies Ricardian Equivalence, then it, thus, doesn't matter whether the government finances government expenditures with taxes or with government debt.

It is easy to see why the model developed in this section satisfies Ricardian Equivalence. Note that if the laws of motion for money supply, $M^s_{t+1}(= M_{t+1})$, and government expenditures, g_t, are taken as given, then the following system can be used to solve for $c(s_t)$, $k(s_t)$, $h(s_t)$, $q\left(s^a_t\right)$, $p\left(s^a_t\right)$, and λ_t.

$$\lambda_t = \frac{\partial u(c_t, l_t)}{\partial c_t} - \frac{\partial u(c_t, l_t)}{\partial l_t}\frac{\partial v(c_t, m_t)}{\partial c_t}, \tag{2.13a}$$

$$\lambda_t = \beta \mathrm{E}\left[\lambda_{t+1}\left(\theta_{t+1}\frac{\partial f(k_{t+1}, h_{t+1})}{\partial k_{t+1}} + 1 - \delta\right)|I_t\right], \tag{2.13b}$$

$$\frac{\partial u(c_t, l_t)}{\partial l_t} = \lambda_t\theta_t\frac{\partial f(k_t, h_t)}{\partial h_t} \tag{2.13c}$$

$$\frac{q_t\lambda_t}{p_t} = \beta \mathrm{E}_t\left[\frac{\lambda_{t+1}}{p_{t+1}}\right], \tag{2.13d}$$

$$\frac{\lambda_t}{p_t} = \beta \mathrm{E}\left[\left(\lambda_{t+1} - \frac{\partial u(c_{t+1}, l_{t+1})}{\partial l_{t+1}}\frac{\partial v(c_{t+1}, m_{t+1})}{\partial m_{t+1}}\right)\frac{1}{p_{t+1}}|I_t\right], \text{and} \tag{2.13e}$$

$$c_t + k_{t+1} + g_t = \theta_t f(k_t, h_t) + (1-\delta)k_t \tag{2.13f}$$

Neither government debt not taxes appear in this system of equations, so the solution is not affected by a change in these variables. The reason for this result is that economic agents realize that a reduction in current taxes caused by an increase in debt financing leads to an increase in future taxes since at some point the debt has to be repaid. The intertemporal budget set for the agent is, thus, not affected by a reduction in current taxes - as long as government expenditures remain the same. Consequently, the optimal choice is not affected either.

Ricardian equivalence implies that the time path of B^s_{t+1} does not affect the agents choices. You might be tempted to say that the supply of government debt is, thus, not a state variable. It is typically better, however, not to think too much about these kind of properties in constructing the set of state variables. Note that the supply of government debt still affect taxes. More importantly, it is worse to miss a state variable then to have a state variable in your model that in your particular model doesn't have an effect.

Neutrality

Again consider the solutions for $c(s_t)$, $k(s_t)$, $h(s_t)$, $\lambda(s_t)$, $p\left(s_t^a\right)$, and $q\left(s_t^a\right)$ that solve the system of equations 2.13 at the exogenously specified values of the money supply. Now take as given the state variables in period τ and suppose that you multiply the money supply in each period by a factor $\phi > 0$ beginning with the beginning-of-period money supply in period τ. Thus the new money supply $\widetilde{M}_{\tau+j} = \phi M_{\tau+j}\ \forall j \geq 0$.[4] Then even without knowing what the particular solutions for this economy look like you can figure out how this change in the money supply will affect the variables in this economy. In fact, after the change in money supply the price level will be equal to the old price level multiplied with a factor ϕ and other variables remain the same. That is, $\widetilde{p}\left(\widetilde{s}_t^a\right) = \phi p(s_t^a)$, $\widetilde{c}(s_t) = c(s_t)$, $\widetilde{k}(s_t) = k(s_t)$, $\widetilde{h}(s_t) = h(s_t)$, and $\widetilde{q}\left(\widetilde{s}_t^a\right) = q(s_t^a)$. It is not hard to see why this is the case. The new solution has to satisfy

$$\widetilde{\lambda}_t = \frac{\partial u(\widetilde{c}_t, \widetilde{l}_t)}{\partial \widetilde{c}_t} - \frac{\partial u(\widetilde{c}_t, \widetilde{l}_t)}{\partial \widetilde{l}_t}\frac{\partial v(\widetilde{c}_t, \widetilde{m}_t)}{\partial \widetilde{c}_t}, \tag{2.14a}$$

$$\widetilde{\lambda}_t = \beta \mathrm{E}\left[\widetilde{\lambda}_{t+1}\left(\widetilde{\theta}_{t+1}\frac{\partial f(\widetilde{k}_{t+1}, \widetilde{h}_{t+1})}{\partial \widetilde{k}_{t+1}} + 1 - \delta\right)|I_t\right], \tag{2.14b}$$

$$\frac{\partial u(\widetilde{c}_t, \widetilde{l}_t)}{\partial \widetilde{l}_t} = \widetilde{\lambda}_t \theta_t \frac{\partial f(\widetilde{k}_t, \widetilde{h}_t)}{\partial \widetilde{h}_t} \tag{2.14c}$$

$$\frac{\widetilde{q}_t \widetilde{\lambda}_t}{\phi p_t} = \beta \mathrm{E}\left[\frac{\widetilde{\lambda}_{t+1}}{\phi p_{t+1}}|I_t\right], \tag{2.14d}$$

$$\frac{\widetilde{\lambda}_t}{\phi p_t} = \beta \mathrm{E}\left[\left(\widetilde{\lambda}_{t+1} - \frac{\partial u(\widetilde{c}_{t+1}, \widetilde{l}_{t+1})}{\partial \widetilde{l}_{t+1}}\frac{\partial v(\widetilde{c}_{t+1}, \widetilde{m}_{t+1})}{\partial \widetilde{m}_{t+1}}\right)\frac{1}{\phi p_{t+1}}|I_t\right], \text{ and} \tag{2.14e}$$

$$\widetilde{c}_t + \widetilde{k}_{t+1} + g_t = \widetilde{\theta}_t f(\widetilde{k}_t, \widetilde{h}_t) + (1-\delta)\widetilde{k}_t \tag{2.14f}$$

where we have already substituted in our guess for $\widetilde{p}_t$. It is easy to see that the factor ϕ cancels out in each equation and you end up with the same set of equations as in 2.14 and, thus, with the same solutions.

2.2.4 Steady-State Solution and Superneutrality

In this section we will make use of the following assumption and lemma.

Condition 1 (functional forms) $U(c_t, l_t) = c_t^{\nu} l_t^{1-\nu}$, $f(k_t, h_t) = k_t^{\alpha} h_t^{1-\alpha}$, $v(c_t, m_t) = \xi\left(c_t\right)^{\frac{1}{1-\kappa}}\left(m_t\right)^{\frac{-\kappa}{1-\kappa}}$, $0 < \nu < 1$, $0 < \kappa < 1$, $\alpha > 0$, *and* $\xi > 0$.

[4] Note that we use the equilibrium condition that money demand equals money supply.

Lemma 2 (equal growth rates) *If $x = y + z$ and the growth rates of all three variables are constant, then the growth rates are equal.*

Before analyzing the full stochastic version of a dynamic model it is often useful to first learn about the properties of the non-stochastic version of the model. The first step would be to replace the stochastic variables with their unconditional means. Let the unconditional mean of M^s_{t+1}/M^s_t be equal to μ and the unconditional mean of g_t be equal to g. Moreover, we assume without any loss of generality that the unconditional mean of θ_t is equal to one and of the unconditional mean of B_{t+1} is equal to zero. We define a *stationary state* as a solution of the model in which all variables are constant and a *steady state* as a solution in which all growth rates are constant.

Suppose that $\mu \neq 0$. The question arises whether real variables like consumption could have non-zero growth rates in an economy in which the growth rate of money is not equal to zero. If the conditions in assumption 2.1 are satisfied, then it is easy to show that such a solution can not be a steady-state solution. In particular, we continue by showing that in a steady-state solution all variables except nominal money balances and prices are constant. The equations for the steady-state version of the competitive equilibrium in 2.13 are given by

$$\lambda_t = \nu \left(\frac{c_t}{l_t}\right)^{\nu-1} + \frac{1-\nu}{1-\kappa}\left(\frac{c_t}{l_t}\right)^{\nu} \xi (c_t/m_t)^{\kappa/(1-\kappa)}, \tag{2.15a}$$

$$1 = \beta \left[\lambda_g \left(\alpha \left(\frac{k_t}{h_t}\right)^{\alpha-1} + 1 - \delta\right)\right], \tag{2.15b}$$

$$(1-\nu)\left(\frac{c_t}{l_t}\right)^{\nu} = \lambda_t (1-\alpha)\left(\frac{k_t}{h_t}\right)^{\alpha}, \tag{2.15c}$$

$$q_t = \beta \left[\frac{\lambda_g}{p_g}\right], \tag{2.15d}$$

$$1 = \beta \left[\frac{\lambda_g}{p_g} + \frac{1}{\lambda_t p_g}\frac{\kappa(1-\nu)}{1-\kappa}\left(\frac{c_t}{l_t}\right)^{1-\nu} \xi (c_t/m_t)^{1/(1-\kappa)}\right], \text{and} \tag{2.15e}$$

$$c_t + k_{t+1} + g_t = k_t^{\alpha} h_t^{1-\alpha} + (1-\delta) k_t, \tag{2.15f}$$

where x_g is equal to x_{t+1}/x_t which by definition of a steady state is constant. Since $h_t + l_t + v_t = 1$ and the right-hand side doesn't grow, lemma 2.2 implies that the variables on the left-hand side should be constant in a steady state too. Then 2.15b immediately tells us that k_t is constant in a steady state as well. Furthermore, 2.15f implies that c_t is constant which in turn implies that m_t is constant (since v_t is constant). If m_t is constant, then the inflation rate has to equal the growth rate of the money supply. Finally, if m_t and c_t are constant, then λ_t and q_t are constant as well and we can rewrite the system of equations in 2.15 as follows.

$$\lambda = \nu \left(\frac{c}{l}\right)^{\nu-1} + \frac{1-\nu}{1-\kappa}\left(\frac{c}{l}\right)^{\nu} \xi(c/m)^{\kappa/(1-\kappa)}, \tag{2.16a}$$

$$1 = \beta\left[\alpha\left(\frac{k}{h}\right)^{\alpha-1} + 1 - \delta\right], \tag{2.16b}$$

$$(1-\nu)\left(\frac{c}{l}\right)^{\nu} = \lambda(1-\alpha)\left(\frac{k}{h}\right)^{\alpha}, \tag{2.16c}$$

$$q = \beta\left[\frac{1}{p_g}\right], \tag{2.16d}$$

$$1 = \beta\left[\frac{1}{p_g} + \frac{1}{\lambda p_g}\frac{\kappa(1-\nu)}{1-\kappa}\left(\frac{c}{l}\right)^{1-\nu} \xi(c/m)^{1/(1-\kappa)}\right], \text{and} \tag{2.16e}$$

$$c + g = k^{\alpha}h^{1-\alpha} - \delta k, \tag{2.16f}$$

We will use the system of equations in 2.16 to analyze how variables change in response to a change in the growth rate of money supply. Recall that if the amount of real money balances is constant in a steady state, then the growth rate of money equals the inflation rate.

We say that a model is *superneutral* if in response to a change in the steady-state growth rate of money supply (or inflation) real variables except possibly real money balances and transfers do not change. We exclude real money balances because in any sensible model, the demand for real money balances depends negatively on the rate of return on money and is, thus, inversely related to an increase in the growth rate of money supply. Similarly a change in real money balances typically changes the level of real taxes.

To show that this model is superneutral we have to find a subsystem with which we can solve for c, k, h, and l that does not contain the money growth rate. For this model this cannot be done, so this model is not superneutral. To understand why suppose to the contrary that c, k, h, and l are not affected by a change in μ. If real money balances change, then equations 2.16a and 2.16c imply that either c, h, or l has to change as well. If the level of real money balances would remain the same then 2.16e implies that λ changes which according to 2.16a implies that either c or l has to change.

The intuition for this lack of superneutrality is the following. Since real money balances are constant in the steady state, we know that an increase in the growth rate of money supply corresponds to an equal increase in the inflation rate. This lowers the real return on holding real money balances and makes it more expensive to hold money. This plays a role in two substitution processes. Note that the agent can use real money balances and shopping time to produce acquisition services. Since real money balances have become relatively more expensive, the economic agent will substitute real money balances for shopping time. The increase in shopping time puts downward pressure on leisure and hours worked. The second substitution process deals with the two arguments in the agent's utility function, consumption and leisure. To acquire consumption the agent needs real money balances but to acquire leisure he doesn't. The increase in inflation, thus, increases the price of consumption relative to the price of leisure. The agent will respond by reducing consumption and increasing leisure. The latter effect puts downward pressure on labor supply. In this economy we can, thus, expect an increase in the steady-state growth rate of money and inflation to reduce economic activity.

2.2.5 Social planner's problem

In the neoclassical growth model developed in Chapter 1, the allocation of the competitive equilibrium coincides with the allocation in the social planner's problem and the competitive equilibrium allocation is, thus, Pareto optimal. In contrast, monetary competitive equilibriums are often not Pareto optimal. To analyze this issue we specify the first-order conditions for the social planner's problem and compare those with the ones obtained above for the competitive equilibrium.[5] It is important to distinguish between the social planner and the government. The social planner is a fictitious agent, while the government is the body of institutions that actually sets monetary and fiscal policy.

The social planner faces the same technology constraints as the agents in the economy. In particular, the social planner also has to combine real money balances and shopping time to acquire consumption services. If there is only one representative agent in the economy then the objective function of the social planner coincides with that of the representative agent. The social planner differs from the actual agents in the model in that the social planner's budget constraint is the overall budget constraint. The social planner's optimization problem is, thus, given by

$$\begin{aligned} \max_{\left\{ \begin{array}{c} c_{t+j}, h_{t+j}, k_{t+j+1}, \\ v_{t+j}, m_{t+j} \end{array} \right\}_{j=0}^{\infty}} \quad & \mathrm{E}\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, 1 - h_{t+j} - v_{t+j}) | \, I_t \right] \\ \text{s.t.} \quad c_{t+j} + k_{t+1+j} + g_{t+j} &= \theta_{t+j} f(k_{t+j}, h_{t+j}) + (1-\delta) k_{t+j} \\ v_{t+j} &= v\left(c_{t+j}, \frac{M_{t+j}}{p_{t+j}} \right) \\ k_t & \text{ predetermined} \end{aligned} \tag{2.17}$$

Note that government debt is not included as a choice variable for the social planner. The reason is that in a model that satisfies Ricardian Equivalence the choice for government debt doesn't affect the utility of the agent. Also, since the p_t is not yet determined in period t, period t real money balances are included as a choice variable for the social planner. The first-order conditions for the

[5] The model analyzed in this chapter does not have markets for capital and labor. One can, however, easily decentralize the model without changing the central argument of this section.

social planner's problem are equal to

$$\lambda_t = \frac{\partial u(c_t, l_t)}{\partial c_t} - \frac{\partial u(c_t, l_t)}{\partial l_t}\frac{\partial v(c_t, m_t)}{\partial c_t}, \tag{2.18a}$$

$$\lambda_t = \beta \mathrm{E}\left[\lambda_{t+1}\left(\theta_{t+1}\frac{\partial f(k_{t+1}, h_{t+1})}{\partial k_{t+1}} + 1 - \delta\right)|I_t\right], \tag{2.18b}$$

$$\frac{\partial u(c_t, l_t)}{\partial l_t} = \frac{\partial u(c_t, l_t)}{\partial c_t}\theta_t\frac{\partial f(k_t, h_t)}{\partial h_t}, \tag{2.18c}$$

$$\frac{\partial u(c_t, l_t)}{\partial l_t}\frac{\partial v(c_t, m_t)}{\partial m_t} = 0, \tag{2.18d}$$

$$c_t + k_{t+1} + g_t = \theta_t f(k_t, h_t) + (1-\delta)k_t, \text{ and} \tag{2.18e}$$

$$\lim_{J\to\infty} E\left[\beta^{J-1}\lambda_J k_{J+1}|I_t\right] = 0. \tag{2.18f}$$

When we compare the equations in 2.18 with the equations in 2.13 then we see that all equations are the same except the first-order condition for money. The social planner's first-order conditions indicate that the agent should be completely satiated with real money balances at the optimum since it doesn't cost the social planner anything to increase the level of real money balances.[6] The individual typically would not pick such a large number of real money balances since for every unit of real money balances held he has to pay the opportunity costs, that is, he foregoes interest payments that he could have earned on bond purchases. There are circumstances when the competitive equilibrium does coincide with the social planner's problem. A necessary condition would be that the interest rate is equal to zero in each period (or q_t is equal to one). In that case the opportunity costs of holding money would be equal to zero for the individual agent as well.

Implications for steady-state inflation

If the interest rate is equal to zero then the steady-state inflation rate $\pi = p_{t+1}/p_t - 1$ is equal to $\beta - 1$, which equals (approximately) the negative of the discount rate. This is the famous "Chicago Rule".[7] To understand this optimality result a little bit better recall that the agent uses real money balances and shopping time to produce acquisition services. From the individual's point of view both real money balances and shopping time are costly inputs. From the social planner's point of view, however, real money balances are free and shopping time is costly. Only if the nominal interest rate is equal to zero are the (opportunity) costs of holding real money balances for the individual also equal to zero.

[6] Note that the level of real money balances that satisfies equation 2.18d is infinite.
[7] See, for example, Friedman (1969).

2.2.6 Cash-in-Advance Models

In this section we consider a special case of the shopping-time technology described above. In particular, if we assume that $\bar{\xi} = 1$ and $\kappa = 1$, then shopping time is not productive in acquiring consumption and for every additional dollar of consumption you have to hold one additional dollar of your wealth in the form of money. In addition, we assume that any increase in the money supply during period t can be used to acquire consumption commodities. That is, we have the following constraint.

$$c_t \leq \frac{M_t + (M_{t+1}^s - M_t^s)}{p_t}$$

Before we write down the optimization problem of the agent, it might be useful to give an intuitive description of the sequence of events in each period. At the beginning of the period, the agents observe the realizations of θ_t and ε_t^M. Using beginning-of-period nominal money holdings and any possible money transfer received from the government the agent buys consumption. After shopping the agent returns to the household with the remainder of his money balances, $M_t + (M_{t+1}^s - M_t^s) - p_t c_t \geq 0$. At this point the agent decides how much labor to supply and how much to invest in capital, one-period bonds, and money holdings.

The agent's optimization problem in the cash-in-advance economy is given by

$$\max_{\{C_{t+j}, h_{t+j}, k_{t+j+1}, M_{t+j+1}, B_{t+j+1}\}_{j=0}^{\infty}} E\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, 1-h_t) |\, I_t\right]$$

$$\begin{aligned}
\text{s.t.} \quad & k_{t+1+j} + \tfrac{M_{t+1+j}}{p_{t+j}} + q_{t+j}\tfrac{B_{t+1+j}}{p_{t+j}} + \tau_{t+j} = \theta_{t+j} f(k_{t+j}, h_{t+j}) \\
& +(1-\delta)k_{t+j} + \left(\tfrac{M_{t+j} + (M_{t+j+1}^s - M_{t+j}^s)}{p_{t+j}} - c_{t+j}\right) + \tfrac{B_{t+j}}{p_{t+j}} \\
& c_{t+j} \leq \tfrac{M_{t+j} + (M_{t+j+1}^s - M_{t+j}^s)}{p_{t+j}} \\
& k_t, M_t, \text{ and } B_t \text{ predetermined}
\end{aligned} \tag{2.19}$$

The first-order conditions for this problem are given by

$$\frac{\partial u(c_t, l_t)}{\partial c_t} = \lambda_t + \eta_t, \tag{2.20a}$$

$$\lambda_t = \beta \mathrm{E}\left[\lambda_{t+1}\left(\theta_{t+1}\frac{\partial f(k_{t+1}, h_{t+1})}{\partial k_{t+1}} + 1 - \delta\right) |I_t\right], \tag{2.20b}$$

$$\frac{\partial u(c_t, l_t)}{\partial l_t} = \lambda_t \theta_t \frac{\partial f(k_t, h_t)}{\partial h_t} \tag{2.20c}$$

$$\frac{q_t \lambda_t}{p_t} = \beta \mathrm{E}\left[\frac{\lambda_{t+1}}{p_{t+1}} |I_t\right], \tag{2.20d}$$

$$\frac{\lambda_t}{p_t} = \beta \mathrm{E}\left[\frac{\lambda_{t+1} + \eta_{t+1}}{p_{t+1}} |I_t\right], \tag{2.20e}$$

$$k_{t+1} + \frac{M_{t+1}}{p_t} + q_t \frac{B_{t+1}}{p_t} + \tau_t = \theta_t f(k_t, h_t) + \tag{2.20f}$$

$$(1-\delta)k_t + \left(\frac{M_t + (M^s_{t+1} - M^s_t)}{p_t} - c_t \right) + \frac{B_t}{p_t},$$

$$c_t \leq \frac{M_t + (M^s_{t+1} - M^s_t)}{p_t}, \tag{2.20g}$$

$$\eta_t \left(\frac{M_t - (M^s_{t+1} - M^s_t)}{p_t} - c_t \right) = 0 \tag{2.20h}$$

$$\eta_t \geq 0 \tag{2.20i}$$

$$\lim_{J \to \infty} \beta^{J-1} \mathrm{E} \left[\lambda_J k_{J+1} | I_t \right] = 0, \tag{2.20j}$$

$$\lim_{J \to \infty} \beta^{J-1} \mathrm{E} \left[q_J \frac{\lambda_J}{p_J} B_{J+1} | I_t \right] = 0 \text{ and} \tag{2.20k}$$

$$\lim_{J \to \infty} \beta^{J-1} \mathrm{E} \left[\frac{\lambda_J}{p_J} M_{J+1} | I_t \right] = 0, \tag{2.20l}$$

Note that the Lagrange multiplier corresponding to the cash-in-advance constraint,η_t, is equal to zero if consumption is strictly less than the amount of real money balances.[8] As in the shopping-time model, the marginal utility of consumption exceeds the marginal utility of wealth. Since the increase of money is now given directly to the shopper, the amount of taxes is given by

$$\tau_t = g_t - \frac{q_t B^s_{t+1} - B^s_t}{p_t}. \tag{2.21}$$

A competitive equilibrium consist of solutions for c_t, h_t, k_{t+1}, M_{t+1}, B_{t+1}, τ_t, λ_t, η_t, p_t, and q_t that satisfy the equations in 2.20 and 2.21 and the following two equilibrium conditions.

$$M_{t+1} = M^s_{t+1} \text{ and} \tag{2.22}$$

$$B_{t+1} = B^s_{t+1}. \tag{2.23}$$

If we combine equations 2.20a, 2.20d, and 2.20e then we get

$$q_t \mathrm{E}_t \frac{1}{p_{t+1}} \frac{\partial u(c_{t+1}, l_{t+1})}{\partial c_{t+1}} = \mathrm{E}_t \left[\frac{\beta}{p_{t+2}} \frac{\partial u(c_{t+2}, l_{t+2})}{\partial c_{t+2}} \right].$$

This first-order equation for bonds is very similar to the first-order equation for bonds in a model without a cash-in-advance constraint. The difference is that there is a shift in timing. The reason is that if there was no cash-in-advance

[8]Strictly speaking we would have to also add the constraint that $\lambda_t \geq 0$ but this non-negativity constraint is never binding for regular functional form specifications.

constraint buying a dollar worth of bonds means reducing consumption with $1/p_t$ units, the value of which is equal to $(\partial U(c_t, l_t)/\partial c_t)/p_t$. In a cash-in-advance economy buying one dollar worth of bonds means giving up one dollar of money holdings in this period and this means giving up consumption in the *next* period.

Note that in period t the agent chooses M_{t+1}, which he needs to buy consumption in period $t+1$. Ideally he wouldn't want to hold anymore money balances than is absolutely necessary since on money balances he doesn't earn any interest and on bonds he does. However, the economic agent doesn't know yet the value of p_{t+1} and the optimal choice for c_{t+1} when he has to choose M_{t+1} So it seems logical that the cash-in-advance constraint is not always binding. For example, if productivity is unexpectedly low in period t then you would expect that the agent would like to consume less than originally planned and have less nominal money balances than needed. Similarly, when the amount of money supply is exceptionally high then you would expect the agent to have excess money balances. Although this intuition would definitely be correct if prices are exogenous, it turns out that this intuition ignores the endogenous response of equilibrium price levels to these kind of shocks.

To understand the last statement better we will consider a version of the cash-in-advance economy developed in this section in which the constraint turns out to be always binding in equilibrium. In particular, suppose that the following assumption holds.

- $U(c_t, l_t) = \ln(c_t) + \delta \ln(l_t)$,
- $M_t^s / M_{t+1}^s < 1/\beta \ \forall t$, and
- $S = \sup m_t/c_t < \infty$.

The condition that $M_t^s/M_{t+1}^s < 1/\beta$ is very weak and even allows for a negative growth rate of money supply as long as it isn't too negative. The condition that $\sup m_t/c_t < \infty$ rules out irregular cases. The following proposition shows that under this condition the cash-in-advance constraint is binding in every period.

Proposition 3 *If Condition 4 holds then $\eta_t > 0 \ \forall t$.*

Proof. *Define S_t as m_t/c_t. Suppose to the contrary that there are states of nature such that the constraint is not binding. Consider a state of nature such that $S_\tau = S - \varepsilon_\tau$, with $\varepsilon_\tau \geq 0$ and $\eta_\tau = 0$. Since S equals $\sup S_t$ we can choose ε_τ to be arbitrarily small. Intuitively, we focus on the state where the constraint is least binding, i.e. m_t/c_t is the highest. The sup is used since the max may not exist. Note, it may be possible that $S_\tau = 1$. Since $\eta_\tau = 0$ we have*

$$\frac{\lambda_\tau}{p_\tau} = \frac{1}{p_\tau c_\tau}.$$

Combining this equation with 2.20a and 2.20e gives

$$\frac{\lambda_\tau}{p_\tau} = \beta E\left[\frac{\lambda_{\tau+1} + \eta_{\tau+1}}{p_{\tau+1}} | I_\tau\right] = \beta E\left[\frac{1}{p_{\tau+1} c_{\tau+1}} | I_\tau\right] \tag{2.24}$$

or

$$1 = \beta E\left[\frac{p_\tau c_\tau}{p_{\tau+1}c_{\tau+1}}|I_\tau\right]. \tag{2.25}$$

Thus

$$1 = \beta E\left[\frac{M_\tau/S_\tau}{M_{\tau+1}/S_{\tau+1}}|I_\tau\right] = \beta E\left[\frac{M_\tau}{M_{\tau+1}}\frac{S_{\tau+1}}{S-\varepsilon_\tau}|I_\tau\right] \leq \beta E\left[\frac{M_\tau}{M_{\tau+1}}|I_\tau\right],$$

where the inequality follows from the definition of S. But the assumption made about money growth contradicts that $1 \leq \beta E\left[M_t/M_{t+1}|I_t\right] = \beta M_t/M_{t+1}$. ■

2.3 Overlapping-Generations Models

In the type of models developed in the last section, agents hold money because it is either assumed that real money balances are an essential input to obtain consumption or it is assumed that holding wealth in the form of money gives utility that other forms of wealth do not provide. In such models money always has value. In the modern age we use paper money, which except for those that use cocaine, has no intrinsic value; Money only has value because other agents are willing to accept money in exchange for commodities that do have intrinsic value. If you do not expect other agents to accept money, the rational thing for you to do is not to accept money either. Such an equilibrium does not exist in MIU models and this is a drawback of these type of models. In this section, we consider overlapping-generations or OLG models in which equilibria where money has positive value may occur but the case where money is not valued is always an equilibrium too.

An important concept in studying overlapping-generations model is the idea of overaccumulation of capital. In Section 2.3.1, we will show this can never happen in the model of Chapter 1 with infinitely-lived agents. In Section 2.3.2, we lay out the basic overlapping-generations model, and in the last section we consider monetary equilibria in overlapping-generations models.

2.3.1 Overaccumulation of Capital in Infinite-Horizon Models

Consider again the non-stochastic version of the model developed in Chapter 1.

$$\begin{array}{c} \max_{\{c_t,k_{t+1}\}_{t=1}^\infty} \sum_{t=1}^\infty \beta^{t-1}\ln(c_t) \\ \text{s.t. } c_t + k_{t+1} \leq k_t^\alpha + (1-\delta)k_t \\ k_{t+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{2.26}$$

We have adopted a logarithmic current-period utility function but the results in this section are true for more general utility functions as well. The first-order condition for this problem is given by

$$\begin{array}{c} \left(k_t^\alpha + (1-\delta)k_t - k_{t+1}\right)^{-1} = \\ = \beta\left[\left(k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2}\right)^{-1}\left\{\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right\}\right] \end{array} \tag{2.27}$$

and the expression for the steady-state value for capital, k^{ss}, is the following:

$$k^{ss} = \left(\frac{1 - \beta(1-\delta)}{\alpha\beta} \right)^{\frac{1}{\alpha-1}} \tag{2.28}$$

It can be shown that the time path of capital that is the solution to 2.26 converges to k^{ss}. Now consider the following static maximization problem:

$$\begin{array}{c} \max_{\{c,k\}} \ln(c) \\ \text{s.t. } c + k \leq k^{\alpha} + (1-\delta)k \end{array} \tag{2.29}$$

Note that this problem chooses the constant or steady-state values of capital and consumption with which the agent would obtain the highest possible current-period utility level, which of course correspond to choosing the highest possible (constant) consumption value. The first-order condition for this problem is

$$\alpha k^{\alpha-1} - \delta = 0$$

and the capital stock that solves this problem is called the *golden-rule* capital stock and is equal to

$$k^{gr} = \left(\frac{\delta}{\alpha} \right)^{\frac{1}{\alpha-1}}. \tag{2.30}$$

Whenever the capital stock is bigger than the golden-rule capital stock then the marginal productivity of capital is less than the depreciation rate, that is, the net return on capital is negative. It is important to understand that the maximization problem in 2.29 is only introduced to introduce the concept of overaccumulation of capital and to understand the actual optimization problem in 2.26 better. We are not saying that 2.29 actually is relevant for any agent's behavior.

For any positive initial capital stock, capital will converge monotonically towards k^{ss}. Thus, if $k_1 > k^{ss}$ then $k_1 > k_2 > k_3 > \cdots > k^{ss}$ and if $k_1 < k^{ss}$ then $k_1 < k_2 < k_3 < \cdots < k^{ss}$. Now suppose that $k_1 \neq k^{ss}$ and consider the time path for capital such that $k_t = k_1$ for $t = 2, 3, \cdots$.[9] One way to prove that this investment plan is not optimal is to show that it doesn't satisfy 2.27. But when $k_1 > k^{gr}$ there is also a very intuitive reason why you would never want to keep capital constant at the initial level. The reason is that by setting $k_t = k^{gr} < k_1$ for $k = 2, 3, \cdots$ the agent would have both a higher consumption level in period 1, since his investment level is smaller, and a higher consumption level thereafter since the highest possible steady-state consumption level is associated with k^{gr}.

It is probably worthwhile to think through why setting $k_t = k_1$ for $t = 2, 3, \cdots$ is also not optimal when $k_1 = k^{gr}$. You might think that this capital path is optimal since it has the highest possible level of steady-state consumption and you don't have to make any additional net investment to get to this high level of capital. If the agent chooses a capital path that converges towards k^{ss}

[9] This will be feasible as long as $k_1 < \delta^{1/(\alpha-1)}$, which exceeds k^{gr} as long as $\alpha < 1$.

then his consumption level will converge towards a level that is lower than the consumption level associated with the golden-state capital stock. By lowering the capital stock below k^{ss}, however, the agent can at least initially enjoy a consumption level that exceeds the golden-rule consumption level which is more important than a lower consumption level in the limit because of discounting.[10]

2.3.2 Non-Monetary Overlapping-Generations Models

In this section, we will develop a very simple overlapping-generations model in which each agent lives for exactly two periods. That is, in every period t a generation of "young" agents is born. In period $t+1$ the generation born in period t becomes "old" and a new generation of young agents is born. We will start by formulating the basic model and discuss Pareto optimality and overaccumulation of capital in non-monetary overlapping-generations models. This discussion will be useful in the next section where we discuss monetary overlapping-generations models.

The basic OLG model

We will start with an OLG model without population growth[11] in which each young agent is endowed with one unit of the consumption commodity. The optimization problem of a young agent would then be the following:

$$\begin{aligned} \max_{c_t^y, c_{t+1}^o, s_{t+1}} \quad & U(c_t^y, c_{t+1}^o) \\ \text{s.t.} \quad & c_t^y + s_{t+1} = 1, \\ & c_{t+1}^o = (1+r_{t+1}) s_{t+1}, \end{aligned} \tag{2.31}$$

where c_t^y is the consumption of the young in period t, c_{t+1}^o is the consumption of the old in period $t+1$, s_{t+1}. The amount saved by the young in period t, and r_{t+1} is the rate of return on savings made in period t. Let $v(c_t^y, c_{t+1}^o)$ denote the marginal rate of substitution. That is

$$v(c_t^y, c_{t+1}^o) = \frac{\partial U(c_t^y, c_{t+1}^o)/\partial c_t^y}{\partial U(c_t^y, c_{t+1}^o)/\partial c_{t+1}^o}.$$

As stated in the following assumption, we assume that the utility function has standard properties.

- $\partial U(c_t^y, c_{t+1}^o)/\partial c_t^y) > 0$, $\partial U(c_t^y, c_{t+1}^o)/\partial c_{t+1}^o > 0$,
- Both consumption commodities are normal goods,
- $v(c_t^y, c_{t+1}^o)$ is continuous,

[10]Note that if $\beta = 1$ the golden-rule capital stock coincides with the steady-state capital stock of the infinite-horizon optimization problem.

[11]Because there is no population growth, the number of young agents is equal to the number of old agents.

- $\lim_{c_t^y \to 0} v(c_t^y, c_{t+1}^o) = \infty$, and
- $\lim_{c_{t+1}^o \to 0} v(c_t^y, c_{t+1}^o) = 0$.

The first-order condition for this problem is given by

$$\frac{\partial U(c_t^y, c_{t+1}^o)}{c_t^y} = \frac{\partial U(c_t^y, c_{t+1}^o)}{c_{t+1}^o}(1 + r_{t+1}). \tag{2.32}$$

First, consider the case where there is no storage technology. This implies that there is no possibility for the young to save for old age at all. It is important to understand that the presence of a bond market wouldn't help. All young agents want to buy bonds so the young cannot buy from other young. For sure, some sneaky old guys would be willing to sell bonds to the young, but the young wouldn't be willing to buy from the old because the old won't be around to pay back when the bonds mature. In equilibrium agents, thus, cannot save and the equilibrium allocation for consumption is one unit when young and zero when old. Such a competitive equilibrium in which no trade occurs is called autarky.

Optimality of the competitive equilibrium

The autarky equilibrium is clearly not a Pareto optimum for regular utility functions.[12] To see why note that the young clearly would be willing to give up ε units of consumption when young for ε units of consumption when old when ε is small.[13] It doesn't happen in a competitive equilibrium, however, because there is no storage and no bond market that can implement this trade. But this transfer is feasible for this economy. In particular, it simply requires taking ε units of the young each period and giving them to the old. The current young then give up ε this period and will receive ε when old from the next generation. Moreover, implementation of such a transfer would generate an additional bonus for this economy since in the period of the initial transfer there are an extra ε units available. They either could be given to the old, who didn't give up any commodities when young, or to the young, who already receive ε units when old, or they could divide the ε units.

A classic article on overlapping generations is Shell (1971). In this article the author makes clear that the competitive equilibrium in this type of overlapping-generations model is not Pareto optimal because of a double infinity. That is, an infinite number of dated commodities and an infinite number of (finite-lived) individuals. Note that if the economy would end in period T then the transfer scheme would not be Pareto improving since the young born in period T would be made worse off.

[12] Note that the marginal utility of consumption when old would be infinite in autarky for regular utility functions.

[13] For regular utility functions the argument would go through as long as ε isn't too large. The value of ε clearly doesn't have to be close to zero.

Now suppose that there is a storage technology available. In particular, suppose that each unit stored when young in period t gives $1+r$ units of consumption when old with $1+r>0$. There will still be no trade between agents in the competitive equilibrium. But by putting commodities in storage when young, the consumption when old will be positive. Note that agents in this economy would want to save even when $r<0$. Whenever $r<0$, however, the competitive equilibrium is not Pareto optimal. Suppose that the young save ξ units when young when the rate of return on savings is negative. Clearly everybody would be better off if a transfer scheme would be implemented where the young give ξ units to the old each period. Under this transfer scheme the young will receive ξ units when old, which is larger than $(1+r)\xi$ units, the amount they earn by using the private storage technology.

Overaccumulation of capital

The competitive equilibrium described above with $r<0$ is similar to the overaccumulation of capital case describe in Section 2.3.1. In both cases the net return on capital is less than zero. The big difference, however, is that in an overlapping-generations model overaccumulation of capital might actually occur in equilibrium, while in the model of Chapter 1 with infinitely-lived agents it never does. The possibility of overaccumulation of capital in the OLG model described above is not due to the fact that the rate of return is fixed. You might think that in a model with a variable marginal product of capital, agents that are faced with a negative rate of return on capital would lower the capital stock and increase the marginal rate of return on savings until it becomes positive. We will now show that this is not necessarily the case. Suppose the optimization of the young is given by

$$\begin{gathered}\max_{c_t^y, c_{t+1}^o, k_{t+1}} U(c_t^y, c_{t+1}^o) \\ \text{s.t.} \quad c_t^y + k_{t+1} = 1 \\ c_{t+1}^o = k_{t+1}^\alpha + (1-\delta)k_{t+1}\end{gathered} \tag{2.33}$$

where k_{t+1} is the capital investment of the young in period t. The first-order condition for this problem is given by

$$\frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_t^y} = \frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_{t+1}^o}\left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right). \tag{2.34}$$

Now let's compare the stationary-state version of this equation

$$\frac{\partial U(c^y, c^o)}{\partial c^y} = \frac{\partial U(c^y, c^o)}{\partial c^o}\left(\alpha k^{\alpha-1} + 1 - \delta\right) \tag{2.35}$$

with the stationary-state version of the first-order equation of the model in Section 2.3.1

$$\frac{\partial U(c)}{\partial c} = \left(\alpha k^{\alpha-1} + 1 - \delta\right)\beta\frac{\partial U(c)}{\partial c}\text{or} \tag{2.36}$$

$$1 = \left(\alpha k^{\alpha-1} + 1 - \delta\right)\beta. \tag{2.37}$$

In the model with infinitely-lived agents, the value of $\partial U(c_t)/\partial c_t$ is equal to $\partial U(c_{t+1})/\partial c_{t+1}$ in a stationary state. This ensures that the stationary-state capital stock is less than the golden-rule capital stock for any utility function. In an overlapping-generations model it is not true in general that $\partial U(c_t^y, c_{t+1}^o)/\partial c_t^y$ equals $\partial U(c_t^y, c_{t+1}^o)/\partial c_{t+1}^o$. This is even true when the utility function would be additively separable, that is when $U(c_t^y, c_{t+1}^o) = u(c_t^y) + \beta u(c_{t+1}^o)$, since c_t^y does not have to be equal to c_{t+1}^o.[14]

Population growth

Above, we mentioned that investing in the storage technology is like overaccumulation when the net return, r, is less than zero. If population growth, n, is not equal to zero, then we have to tighten this statement. In the presence of population growth using the storage technology is a silly thing to do whenever $r < n$ and we say that an economy with positive investment levels when $r < n$ is characterized by overaccumulation of capital. More formally, any competitive equilibrium in which agents save at a rate $r < n$ is not Pareto optimal. The reason is that by using transfers from the young to the old instead of the young saving for their old age themselves, one can make at least one generation better off while making no other generation worse off. Note that the transfer scheme is more attractive when $n > 0$, which means that overaccumulation of capital is more likely to happen with positive population growth.

2.3.3 Monetary Overlapping-Generations Models[15]

In this section we will introduce fiat-money into the model. In contrast to the money-in-the-utility and cash-in-advance models considered in Section 2.2, monetary OLG models do not rely on the assumption that money has intrinsic value or is a necessary input to acquire consumption. Agents are only willing to accept money for commodities, because they expect other agents in turn to accept money for commodities. Unlike the models in 2.2, therefore, overlapping-generations models with fiat money typically have an equilibrium in which money has no value. That is, if agents expect other agents not to accept money, they will not accept it either. This immediately implies that if money is know not to have value at any future date T, it will have no value at any date before T either. We will start considering the economy without storage and then continue by analyzing the case with storage.

Again, we will consider population growth. The population grows at rate n and without loss of generality, we assume that $N_0 = 1$. Thus,

$$N_t = (1+n)^t.$$

[14] See exercise 2.2.

[15] Several of the results in this section are from Wallis (1980).

An OLG model with money and without storage

Let M_t^d be the demand for *end-of-period* nominal units of money. As in 2.31 we assume that the young obtain an endowment of one unit. The fraction of the unit that the young don't consume, $1 - c_t^y$, they can sell at a price p_t in exchange for money. End-of-period t nominal money balances, therefore, are equal to $p_t(1 - c_t^y)$. In the next period, $t+1$, the young will be the old and they can use these money balances together with a monetary lump-sum transfer from the government, T_{t+1}, to buy consumption c_{t+1}^o. The optimization problem of the young born in period t is thus given by

$$\begin{aligned} \max_{c_t^y, c_{t+1}^o, M_t^d} \; & u(c_t^y, c_{t+1}^o) \\ \text{s.t.} \quad M_t^d &= p_t(1 - c_t^y) \\ p_{t+1} c_{t+1}^o &= M_t^d + T_{t+1} \end{aligned} \tag{2.38}$$

The first-order conditions for this problem consist of the two budget constraints and the following Euler equation:

$$\frac{\partial u(c_t^y, c_{t+1}^o)}{\partial c_t^y} = \frac{\partial u(c_t^y, c_{t+1}^o)}{\partial c_{t+1}^o} \frac{p_t}{p_{t+1}} \tag{2.39}$$

In period t there are N_{t-1} old agents and the transfer they get is equal to T_t. This is financed out of the increase in the aggregate money supply, $M_t^s - M_{t-1}^s$.[16] The budget constraint of the government specifies that the increase in nominal money balances is equal to the monetary transfer. That is,

$$M_t^s - M_{t-1}^s = (1+n)^{t-1} T_t. \tag{2.40}$$

The equilibrium condition that aggregate money supply is equal to aggregate money demand can then be written as

$$M_t^s = (1+n)^t M_t^d. \tag{2.41}$$

Note that this equilibrium on the money market implies equilibrium on the commodities market. That is

$$(1+n)^t c_t^y + (1+n)^{t-1} c_t^o = (1+n)^t \times 1 \text{ or} \tag{2.42}$$

$$(1+n) c_t^y + c_t^o = (1+n) \times 1 \tag{2.43}$$

This is, of course, a version of Walras Law. That is, if the young agents demand all the units of nominal money in the possession of the old agents, then the amount of commodities saved by the young got to be equal to the consumption of the old.

If we assume that money supply grows at a constant rate μ and population grows at rate n then the equilibrium condition can be written as

[16] Note that M_t^s is *aggregate* money supply and M_t^d is individual money demand. Both are end-of-period t quantities.

$$(1+\mu)^t M_0^s = (1+n)^t M_t^d. \tag{2.44}$$

Before we analyze a monetary equilibrium, that is, an equilibrium in which money has value we want to repeat the point made in the introduction that this model does have an equilibrium in which money has no value, that is, an equilibrium in which the price level is infinite. In that case, the young would consume their endowment and the old would consume nothing.

Next we will analyze a steady-state solution of the model, that is, we assume that p_t/p_{t+1}=p_{t+1}/p_{t+2}.[17] From 2.38 and 2.40 it follows that real money demand, $L_t = M_t/p_t$, is a function of just the inflation rate. In a steady state, this means that L_t=$L(p_t/p_{t+1})$=$L(p_{t+1}/p_{t+2}) = L_{t+1}$. Combining this with the equilibrium condition and the law of motion for money supply gives

$$1 = \frac{L_t}{L_{t+1}} = \frac{M_t^d/p_t}{M_{t+1}^d/p_{t+1}} = \frac{\frac{(1+\mu)^t M_0^s}{(1+n)^t}}{\frac{(1+\mu)^{t+1} M_0^s}{(1+n)^{t+1}}} \frac{p_{t+1}}{p_t} = \frac{(1+n)}{(1+\mu)} \frac{p_{t+1}}{p_t} \tag{2.45}$$

This implies that in a steady state

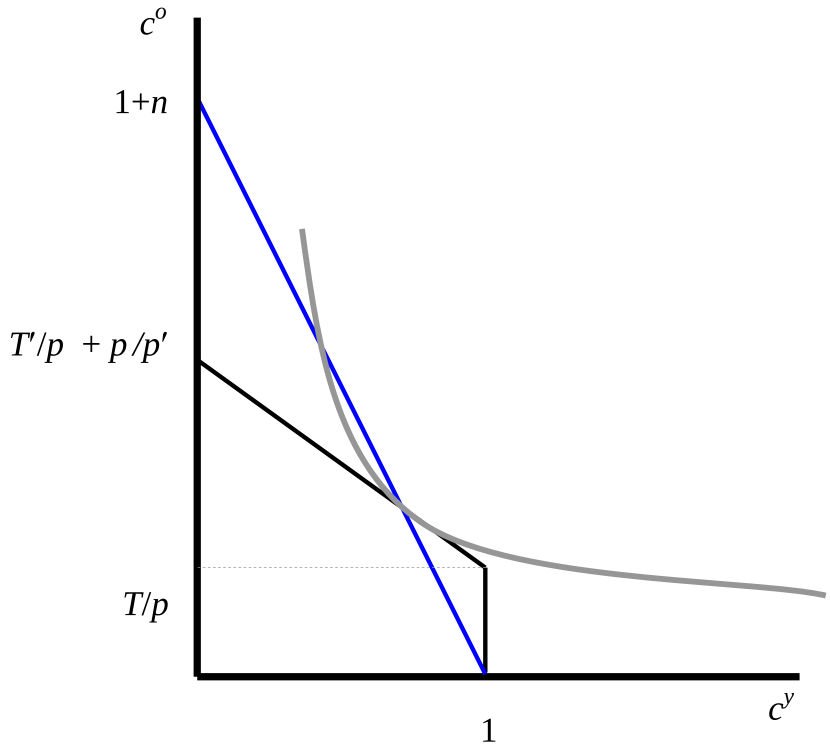
$$\frac{p_t}{p_{t+1}} = \frac{1+n}{1+\mu}. \tag{2.46}$$

In Figure 2.1, we have graphically represented a steady-state monetary equilibrium for the case when $u > 0$.[18] The graph plots the agent's budget constraint that represents the possible choices of the consumption when young, c_t^y, and the consumption when old of the same generation, c_{t+1}^o, which has a slope of $-p_t/p_{t+1}$ and the societies budget constraint that represents the possible choices of the consumption of the young, c_t^y, and the consumption of the old *in the same period*, c_t^o, which has a slope of $-(1+n)$. We can plot both in the same graph, since in the steady state consumption levels are constant. Optimizing behavior implies that the agent chooses an element on his intertemporal budget constraint that is tangent to an indifference curve. At the equilibrium price level this point is feasible, that is, is an element of society's budget constraint. Note that a change in the price level adjusts the real value of the transfer that the old receive. Suppose that the agent's optimal demand for consumption when young and when old is above society's budget constraint. In that case the price level is too low. An increase in the price level will reduce the value of T_{t+1}/p_t. This will cause the budget constraint to shift downward and (for regular preferences) decrease the demand for consumption. Figure 2.2 plots the monetary equilibrium for the case when $u < 0$. Note that in this case the old have to pay a monetary tax.

[17] Unlike the models discussed in Chapter 1 and 2.2, this model could reach the steady state instantaneously.

[18] Without further restrictions it may very well be the case that other equilibria exist as well even with the growth rate of money supply being constant.

Figure 2.1: Monetary equilibrium with $\mu > 0$



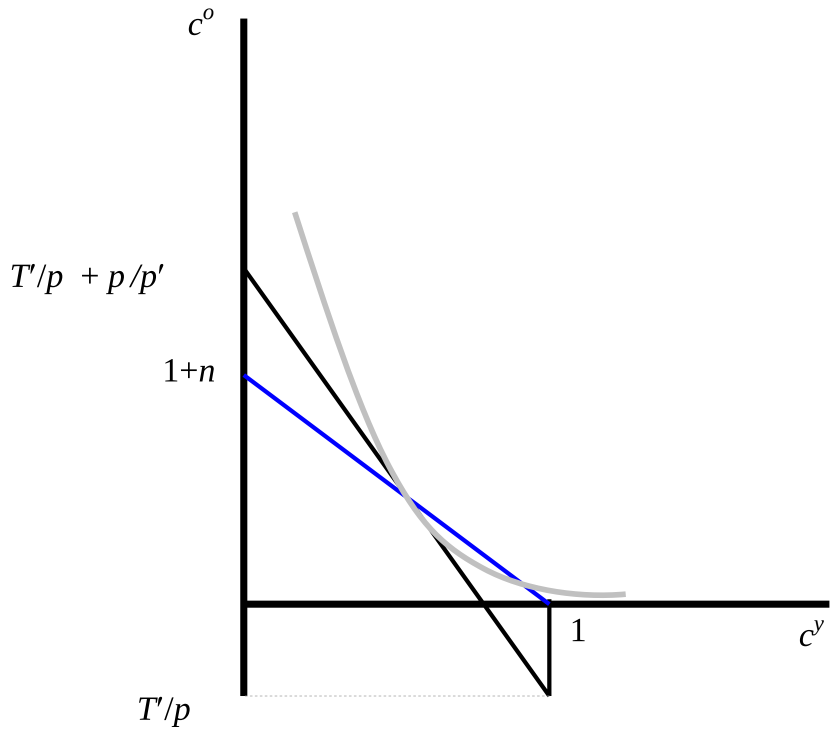
Social planner's problem

Formulating the social planner's problem is a little bit trickier in an OLG model than in the model with an infinitely-lived representative agent, since in an OLG model there are different types of agents and we have to address the issue how to formulate the social planner's objective function. One natural choice would be to give each generation equal weight. In that case the social planner's problem can be written as follows:[19]

$$\begin{array}{rl} \max\limits_{\{c_t^y, c_t^o\}_{t=1}^{\infty}} & u(c_0^y, c_1^o) + \sum_{t=1}^{\infty} u(c_t^y, c_{t+1}^o) \\ \text{s.t.} & (1+n)c_t^y + c_t^o = 1+n \end{array} \tag{2.47}$$

[19]Note that c_0^y is taken as given in the optimization problem.

Figure 2.2: Monetary equilibrium with $\mu < 0$



The Euler equation for this problem is given by

$$(1+n)\frac{\partial u(c^y_{t-1}, c^o_t)}{\partial c^o_t} = \frac{\partial u(c^y_t, c^o_{t+1})}{\partial c^y_t}. \tag{2.48}$$

It is important to realize that besides this particular social planner's solution, there are many other Pareto optimal allocation. When we restrict ourselves to allocations that are constant over time, we can be a little bit more specific. Let $\widehat{c}^y$ and $\widehat{c}^o$ be the steady-state values of consumption that solve 2.48 and society's budget constraint.. All feasible allocations with $c^o \geq \widehat{c}^o$ would be Parteto optimal allocations. Lowering c^o to $\widehat{c}^o$ would make the current old strictly worse off. In contrast, any feasible allocation with $c^o \leq \widehat{c}^o$ and $c^y \geq \widehat{c}^y$ would not be Pareto optimal, since one can make the current young as well as future generations better of by reducing the value of consumption when young to $\widehat{c}^y$ and raising their consumption when old to $\widehat{c}^o$. In the first-period, the reduction of the consumption of the young can be allocated to make, for example, also the current old better off. We will use this property below.

Note that in the social planner's problem the optimality condition equates the marginal utility of c_t^y with the marginal utility of c_t^o (appropriately weighted with the population growth rate), while the optimality condition for the individual's problem equates the marginal utility of c_t^y with the marginal utility of c_{t+1}^o (appropriately weighted with the inflation rate). When we focus on steady-state solutions, however, the timing difference doesn't matter and we can compare the social planner's solution with the solution of the competitive equilibrium. In particular, the allocation in the monetary competitive equilibrium coincides with the social planner's solution if the first-order condition and the budget constraint coincide, which happens if

$$\frac{p_t}{p_{t+1}} = (1+n). \tag{2.49}$$

Since $1 + \mu = (1 + n)p_{t+1}/p_t$, the allocation of the competitive equilibrium coincides with the social planner's solution if $\mu = 0$. The optimal (gross) rate rate of inflation in this model is, thus, equal to $1/(1+n)$. You might think that this optimal rate of inflation differs in an important way from the Chicago rule which stipulates a steady-state deflation rate equal to the rate of time preference. This is not the case, however. The idea behind the Chicago rule is that agents should not economize on holding real money balances, which requires a rate of return on real money balances that is equal to the rate of return on alternative assets, i.e., bonds. In the model of Section 2.2 with infinitely-lived agents, the steady-state rate of return on bonds is equal to the rate of time preference and optimality then requires a deflation rate equal to the rate of time preference. Here a similar condition holds. That is, the rate of return on money has to equal society's rate of return, which is equal to the population growth rate.

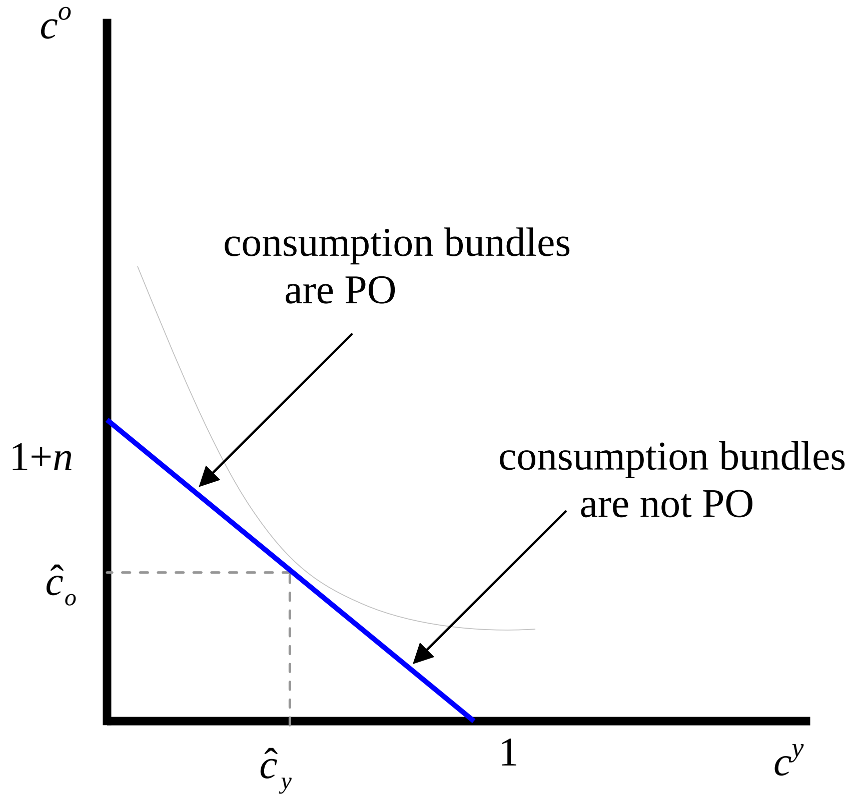
The allocation in the monetary equilibrium is Pareto optimal when $\mu = 0$. The following proposition establishes the results for the case when $\mu \neq 0$.

Proposition 4 *If $\mu > 0$ the steady-state monetary equilibrium is not Pareto optimal and if $\mu \leq 0$ the steady-state monetary equilibrium is Pareto optimal.*

The proof of this proposition is fairly intuitive. First consider the case when $\mu > 0$, which is graphically documented in Figure 2.1. Start with the case in which $\mu = 0$ and the budget constraint of the individual, thus, coincides with the budget constraint of the population. Now move towards a situation with $\mu > 0$. Then two things happen with the budget constraint. First, there is an upward shift in the budget constraint because of the positive nominal transfer to the old. Second, the relative price of c^y falls, since at the higher inflation rate, the real rate of return is lower. Under the assumptions made on preferences this means that the optimal demand for consumption when young should increase. Consequently, when we compare the situation at $\mu > 0$ with how it was at $u = 0$, then the value of c^y will be higher than $\widehat{c}^y$ and the value of c^o less than $\widehat{c}^o$. From the discussion above we know that this is not a Pareto optimal allocation. For the case where $\mu < 0$, the same reasoning can be used to show that the value of c^y will be less than $\widehat{c}^y$ and the value of c^o more than $\widehat{c}^o$. From the discussion above, we know that these are Pareto efficient.

The ideas can be summarizes as follows. Figure 2.3 gives the graphical representation of the social planner problem. The points with $c^o < \widehat{c}^o$ are not optimal because you can move directly to the optimal combination of $\widehat{c}^y$ and $\widehat{c}^o$ and the current old would not mind since their consumption increases. The points with $c^o > \widehat{c}^o$, however, are Pareto optimal. Although the young would like to move towards the optimal combination, it would require lowering consumption of the current old. The discussion in the last paragraph makes clear that $c^o < \widehat{c}^o$, when $\mu > 0$ and that $c^o > \widehat{c}^o$ when $\mu < 0$.

Figure 2.3: Steady state of social planner problem



OLG model with money and storage

The following proposition from Wallace (1980) gives the necessary and sufficient conditions for the existence of a monetary equilibrium.[20]

Proposition 5 (existence of monetary equilibrium) *At least one monetary equilibrium exist if and only if $(1+n)/(1+\mu) \geq 1+r$.*

Proof. *To proof the necessity part assume to the contrary that $(1+r)(1+\mu) > (1+n)$. Then*

$$\frac{p_t}{p_{t+1}} = \frac{M^s_{t+1}}{(1+\mu)M^s_t}\frac{p_t}{p_{t+1}} = \frac{(1+n)M^d_{t+1}}{(1+\mu)M^d_t}\frac{p_t}{p_{t+1}} = \frac{(1+n)m_{t+1}}{(1+u)m_t} \geq 1+r \tag{2.50}$$

where the inequality follows from the fact that for agents to value money, the return on holding money has to be at least as big as the return on the alternative investment. Thus,

$$\frac{m_{t+1}}{m_t} \geq \frac{(1+r)(1+\mu)}{(1+n)}$$

Combining this with the assumption that $(1+r)(1+\mu) > (1+n)$ gives that $m_{t+1}/m_t > 1$ or that real money balances are unbounded. This is impossible, however, since from the budget constraint of the young we know that real money balances are less than 1. ■

Proof. *That the condition is sufficient can be shown by constructing a steady-state monetary equilibrium. The agent's first-order condition can be written as*

$$v(c^y_t, c^o_{t+1}) = \frac{p_t}{p_{t+1}} \text{ or} \tag{2.51}$$

$$v(1-m_t, m_{t+1}(1+n)) = \frac{m_{t+1}}{m_t}\frac{1+n}{1+\mu}. \tag{2.52}$$

It is enough to show that a constant value $m = m_t \ \forall t$ exists such that (i) $m \in (0,1)$, (ii) m satisfies 2.52, and (iii) the return on money exceeds the return on storage. That is,

$$\frac{m_{t+1}}{m_t}\frac{1+n}{1+\upsilon} \geq 1+r. \tag{2.53}$$

This inequality follows directly from the assumption made. Moreover, from the assumption made on the utility function it follows directly that $v(1-m, m(1+n))$ is strictly increasing increasing in m, converges to zero as m goes to zero and goes to infinity as m goes to one. ■

When we consider the expression for steady-state inflation in 2.46 then the condition of the proposition is very intuitive, since it says that the (steady state) rate of return of investing in money investment must be at least as big as the

[20]Note that the conditions are trivially satisfied if there is no storage technology, that is, when $r = -1$.

return on the storage technology. The proposition extends this intuition to the more general case.

We will now discuss the optimality of monetary and non-monetary equilibria in a little bit more detail and distinguish between the case where $n > r$ and the case where $n < r$.

equilibria when n¿r

In this case, a non-monetary equilibrium clearly is not Pareto optimal. If the economy would start using money it can switch to a steady-state allocation that is strictly preferred by at least one generation, since the society's budget set contains the budget set that is generated by the storage technology. Suppose that $n > r$ and that a monetary equilibrium exits. The monetary equilibrium will be Pareto optimal when $\mu \leq 0$, and will not be Pareto optimal when $\mu > 0$. Since in this case the storage technology is not used, it really is the same case as the one without storage, which is covered in Proposition 4. When $n > r$ monetary equilibria can, thus, be both Pareto optimal and not Pareto optimal.

Monetary equilibria when n¡r

In this case, a non-monetary equilibrium clearly is Pareto optimal. Interestingly, a monetary equilibrium—if it exists—is also Pareto optimal.

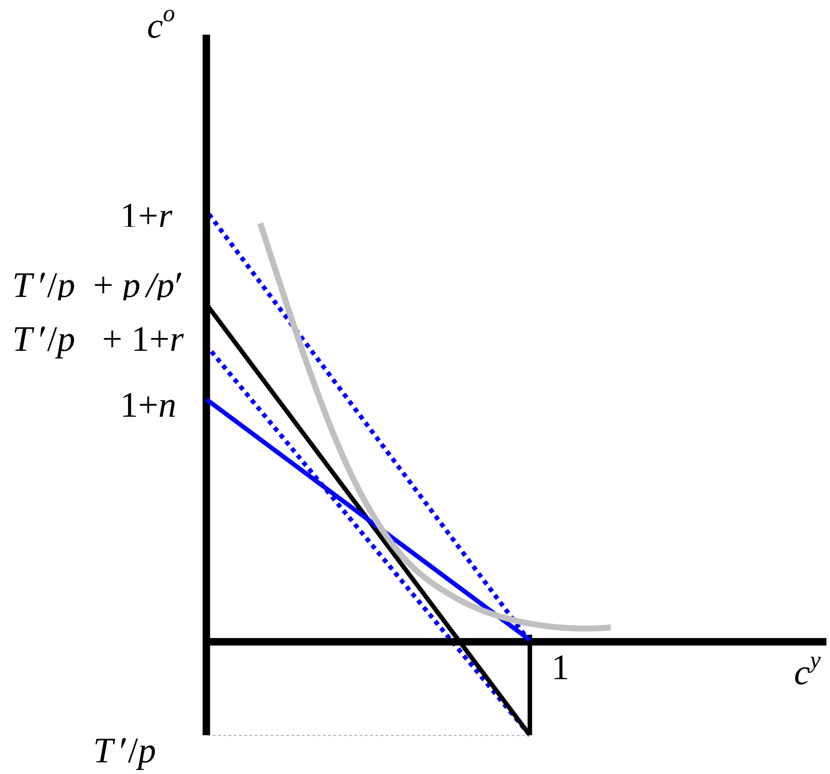
When $n < r$, a monetary equilibrium cannot exist when $u > 0$. This result is fairly intuitive. The steady-state gross rate of return on money is equal to $(1 + n)/(1 + u)$, which can never be bigger than the return on storage when $\mu > 0$ and $r > n$. But a monetary equilibrium can still exist when $\mu \leq 0$. One such equilibrium is represented in Figure 2.4. Note that in the monetary equilibrium the storage technology is not used. Even though the allocation in the monetary equilibrium is on society's budget constraint, which is below the constraint when the economy would use storage, it can be shown that the allocation in the monetary equilibrium is Pareto optimal. The reason is that this economy cannot switch from an economy that uses money to save for old age to an economy that uses storage to save without hurting the generation of the current old, since they are relying on the young to support them.[21] It clearly would have been better if this economy never would have ended up in a monetary equilibrium but given that it did, it cannot start using the more favorable storage technology without hurting some generation.

Note that there is no incentive for individuals to switch to storage either. In this monetary equilibrium the old will be taxed T_{t+1}/p_{t+1} in period $t + 1$ whether they use storage or not. Therefore, the consumption of the old when money is used is higher than the consumption of the old when storage is used,

[21] The proof is in Wallace (1980). It is actually not trivial. For example, you might think that if r is high enough, then the current young can give the current old $c_{o,t+1}$ and simply invest the remainder at r. If r is high enough then the young should still be better off. The flaw in this reasoning is that for such a high r you are violating the condition of the existence of a monetary equilibrium.

that is, $(p_t/p_{t+1})(1-c_t^y)+T_{t+1}/p_{t+1} > (1+r)(1-c_t^y)+T_{t+1}/p_{t+1}$ for any level of c_t^y.

Figure 2.4: Pareto Optimal Monetary Equilibrium with $r > n$



2.4 Exercises

Exercise 2.1: Consider the cash-in-advance economy characterized by the following optimization problem

$$\max_{\left\{\begin{array}{c} c_{t+j}, h_{t+j}, k_{t+1+j}, \\ M_{t+1+j} \end{array}\right\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, 1-h_t) | \, I_t\right]$$

$$\begin{array}{c} \text{s.t.} \quad k_{t+1+j} + \frac{M_{t+1+j}}{p_{t+j}} + \tau_{t+j} = \theta_{t+j} f(k_{t+j}, h_{t+j}) \\ +(1-\delta)k_{t+j} + \left(\frac{M_{t+j}+(M^s_{t+j+1}-M^s_{t+j})}{p_{t+j}} - c_{t+j}\right) \\ c_{t+j} \leq \frac{M_{t+j}+(M^s_{t+j+1}-M^s_{t+j})}{p_{t+j}} \\ k_t \text{ and } M_t \text{ predetermined} \end{array} \tag{2.54}$$

and the following equilibrium condition

$$M_{t+1} = M^s_{t+1}. \tag{2.55}$$

Let

$$\mu_t = M^s_t / M^s_{t+1}. \tag{2.56}$$

The purpose of this question is to show that if we add more noise to the money supply, but keep expected money growth rates the same, only the solution for prices changes. That is, in this cash-in-advance economy, money only has real effects if it changes the expected growth rate of money. In particular, suppose that we introduce a new process for the money growth rate $\overline{\mu}_t = \mu_t \varepsilon_t$, where ε_t is a random variables with mean equal to one and independent of θ_t and μ_t. Thus, $\mathrm{E}\overline{\mu}_t = \mathrm{E}\mu_t$. First, conjecture what prices are under the new law of motion for the money growth rate relative to old prices. Second, show that consumption, capital, and labor supply are not affected.

Exercise 2.2: Consider the optimization problem in 2.26 and consider the constant time path for capital $k_t = \overline{k}$ where $k^{ss} < \overline{k} < k^{gr}$. Show that this time path for capital is not optimal by showing that the agent can increase his life-time utility by reducing his capital stock permanently with ε units.

Exercise 2.3: Suppose that the optimization of the young is given by

$$\begin{array}{c} \max\limits_{c^y_t, c^o_{t+1}, k_{t+1}} \quad u(c^y_t) + \beta u(c^o_{t+1}) \\ \text{s.t.} \quad c^y_t + k_{t+1} = \tau \\ c^o_{t+1} = k^{\alpha}_{t+1} + (1-\delta)k_{t+1} \end{array} \tag{2.57}$$

The utility function $u(\cdot)$ is monotone and strictly concave and satisfies the Inada conditions. Show that overaccumulation will occur if τ is large enough.