1
Dynamic Optimization Problems

1.1 Deriving first-order conditions: Certainty case

We start with an optimizing problem for an economic agent who has to decide each period how to allocate his resources between consumption commodities, which provide instantaneous utility, and capital commodities, which provide production in the next period. At this point we assume that this agent doesn't interact with anybody else in the economy. This might seem strange since the goal is to describe the behavior of macroeconomic variables. There are environments, however, in which the behavior of an economy with a large number of different agents can be described by the optimization problem of a *representative agent.* The assumptions to justify such a representative-agent approach are strong but the relative simplicity makes it the logical starting point. The agent's current period utility function, $u(c_t)$, is assumed to depend only consumption, c_t. The technology that turns capital, k_t, into production is described by the production function $f(k_t)$. Here k_t measures the existing capital stock chosen during $t-1$ and productive at the beginning of period t. Because of depreciation during production, only $(1-\delta)$ will be available during period t. Typically the utility function and the production function are assumed to be continuous, differentiable, strictly increasing and concave in its argument and to satisfy the Inada conditions. A function $g\left(x\right)$ satisfies the Inada conditions if

$$\lim_{x\to 0}\frac{\partial g(x)}{\partial x}=\infty \text{ and } \lim_{x\to\infty}\frac{\partial g(x)}{\partial x}=0.$$

The agent maximizes the following objective function:

$$\begin{array}{c}\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t \\ k_{t+1} \geq 0 \\ k_1 = \overline{k}\end{array} \tag{1.1}$$

The constraints have to hold for every $t = 1, 2, 3, \cdots$. We will refer to this type of dynamic maximiation problem as the sequence problem, because the solution is a sequence. The objective function indicates that the agent lives forever, but he discounts future consumption with the discount factor β.[1] The budget constraint indicates that the price of a capital commodity is equal to the price of one consumption commodity.

The first step in solving this maximization problem is to derive the first-order conditions using the Lagrangian. Before we do this, however, we multiply the period t budget constraint with β^{t-1} and rearrange terms so that the constraint has the standard non-negativity form. This gives

$$\beta^{t-1}\left(f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right) \geq 0. \tag{1.2}$$

This clearly doesn't change the problem, but it makes the interpretation of the Lagrange multiplier somewhat easier. The Lagrangian is given by

$$\begin{array}{c}L(c_1, c_2, \cdots, k_2, k_3, \cdots, \lambda_1, \lambda_2, \cdots) \\ = \sum_{t=1}^{\infty} \beta^{t-1} \left\{u(c_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right]\right\}.\end{array} \tag{1.3}$$

Since there is a budget constraint for each period t, there also is a Lagrange multiplier, λ_t, for every period. Note that we have ignored the non-negativity constraint on capital. For regular infinite-horizon problems, this constraint is never binding. To see why assume that the economic agent organizes a magnificent party in period T, consumes all his period T resources, and sets k_{t+1} equal to zero. For all periods after period T, he wouldn't have any resources and his consumption would be equal to zero. According to the Inada condition, the marginal utility of consumption for a starving agent would be so high that the agent can always improve his utility by reducing his consumption in period T and invest the resources in capital.

The optimization problem above is identical to the following max-min problem:

$$\begin{array}{c}\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \min_{\{\lambda_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) + \\ \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right]\} \\ \text{s.t. } \lambda_t \geq 0 \\ k_1 = \overline{k}\end{array} \tag{1.4}$$

[1] The discount *rate* is equal to $(1-\beta)/\beta$.

Basically, we have replaced the inequality of the budget constraint by the (simpler) non-negativity constraint on λ_t. The first-order conditions for this saddle-point problem are the following equations, which have to hold for all $t \geq 1$:

$$\frac{\partial L(\cdot)}{\partial c_t} = 0 : \frac{\partial u(c_t)}{\partial c_t} = \lambda_t \tag{1.5a}$$

$$\frac{\partial L(\cdot)}{\partial k_{t+1}} = 0 : -\lambda_t + \beta\lambda_{t+1}\left[\frac{\partial f(k_{t+1})}{\partial k_{t+1}} + (1-\delta)\right] = 0 \tag{1.5b}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_t}\lambda_t = 0 : \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right]\lambda_t = 0 \tag{1.5c}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_t} \geq 0 : \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right] \geq 0 \tag{1.5d}$$

$$\lambda_t \geq 0 \tag{1.5e}$$

Since there is a non-negativity restriction on the value of λ_t we have to use the two-part Kuhn-Tucker conditions to derive the first-order conditions associated with the Lagrange Multiplier. If you have trouble deriving these first-order conditions you may want to write out the summation in equation 1.4 for a couple of periods and then differentiate with respect to, for example, c_3, k_4, and λ_3. From equation 1.5c, we see that if the budget constraint is not binding in period t then $\lambda_t = 0$. The period t Lagrange multiplier is equal to the increase in the value of the objective function when the period t budget constraint increased with one unit and, thus, equals the marginal utility of wealth.[2] In this model the marginal utility of wealth is equal to the period t marginal utility of consumption.[3]

Transversality condition.

Because of the infinite dimension of the optimization problem, we also have to consider the transversality condition. For the problem in 1.4 the transversality condition is given by

$$\lim_{T\to\infty} \beta^{T-1}\frac{\partial u(c_T)}{\partial c_T}k_{T+1} = 0. \tag{1.6}$$

To understand the form and the reason for the transversality condition, consider the following finite-period optimization problem:

[2] If we would not have multiplied the budget constraint with β^t, then λ_t would have been equal to $\beta^t\partial u(c_t)/\partial c_t$.

[3] In models where it takes effort to transform wealth into consumption (e.g. because of search costs) then the marginal utility of consumption would be strictly larger than the marginal utility of wealth.

$$\begin{array}{c} \max_{\{c_t,k_{t+1}\}_{t=1}^T} \sum_{t=1}^T \beta^{t-1} u(c_t) \\ \text{s.t. } c_t + k_{t+1} \leq f(k) + (1-\delta)k_t \\ k_{t+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{1.7}$$

The corresponding saddle-point problem is given by

$$\begin{array}{c} \max_{\{c_t,k_{t+1}\}_{t=1}^T} \min_{\{\lambda_t\}_{t=1}^T} \sum_{t=1}^T \beta^{t-1} u(c_t) \\ +\lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}\right]\} \\ \text{s.t. } \lambda_t \geq 0 \\ k_{T+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{1.8}$$

Note that for the last period we do not ignore the non-negativity constraint on capital. If we would ignore this constraint, the economic agent would set k_{T+1} equal to some enormous negative number and consume a lot in period T. The first-order conditions for this problem are identical to those in 1.5 for $t = 1, \cdots, T$, except that the first-order conditions corresponding to k_{T+1} are given by

$$\frac{\partial L(\cdot)}{\partial k_{T+1}} k_{T+1} \quad = \quad 0 : -\beta^{T-1} \lambda_T k_{T+1} = 0 \tag{1.9a}$$

$$\frac{\partial L(\cdot)}{\partial k_{T+1}} \quad \leq \quad 0 : -\lambda_T \leq 0 \tag{1.9b}$$

This equation tells us that solving the constrained optimization problem requires that k_{T+1} has to be set equal to zero unless λ_T is equal to zero, that is, unless the economic agent is completely satiated with consumption. The transversality condition can be obtained by taking the limit of 1.9a as $T \to \infty$. The reason why we may need the transversality condition is that the first-order conditions only determine what is optimal from period to period, but might ignore the overall picture. The transversality condition says that the discounted value of the limiting capital stock cannot be positive. If it would be positive then the agent is building up a capital stock that is too large.

Necessary conditions versus solutions.

Consider the following one-dimensional optimization problem:

$$\max_z h(z), \tag{1.10}$$

where $h(\cdot)$ is a continuous and differentiable function. Then the first-order condition

$$\frac{\partial h(z)}{\partial z} = 0 \tag{1.11}$$

is *not* a solution to the system. The first-order condition only gives a condition that any solution to the system must satisfy. Finding the values for z that satisfy equation 1.11 might be easy or difficult depending on the functional form of $\partial h(z)/\partial z$. Similarly, you have to realize that finding a time path of consumption and capital that satisfies the first-order conditions isn't always that simple. In fact, in most cases we have to rely on numerical procedures to obtain an approximate solution. Furthermore, if z^*solves 1.11 but the second derivative of $h(z)$ at $z = z^*$ is positive then z^* is a minimum and not a maximum.

Showing what the necessary and sufficient conditions are for an infinite-dimensional optimization problem, is not as easy as it is for the one-dimensional optimization problem in 1.10. There are basically three methods to prove that first-order conditions like equations 1.5 are necessary conditions for an optimization problem. Those three methods are (i) calculus of variations,[4] (ii) optimal control, and (iii) dynamic programming. Optimal control requires the weakest assumptions and can, therefore, be used to deal with the most general problems.

Ponzi schemes and transversality conditions.

We now change the problem described above in the following way. Instead of assuming that the agent has a production technology, we assume that each period he receives an endowment y_t and he can smooth his consumption by borrowing and lending at the risk-free rate r. We assume that the interest rate is equal to the discount rate $= (1-\beta)/\beta$. The optimization problem is given by

$$\begin{array}{c} \max_{\{c_t, b_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } c_t + b_{t+1} \leq y_t + (1+r) b_t \\ b_1 = \overline{b} \end{array} \tag{1.12}$$

In this model the agent can save by choosing a positive value for b_{t+1} and he can borrow, and accumulate debt, by choosing a negative value. The first thing to realize is that there is a great opportunity for the economic agent to have a wonderful life. Without any constraint on b_{t+1} he can borrow (and consume) as much as he wants and just pay off the interest payments by borrowing more. These kinds of tricks are called Ponzi-schemes and we have to rule those out by imposing some kind of borrowing constraint. One way to do this is to impose the ad hoc borrowing constraint that $b_t \geq \widetilde{b} < 0$.

[4] A good place to start is to read Section 4.5 in Stockey and Lucas (1989) where it is shown that the first-order conditions and the transversality condition for the standard growth model are sufficient conditions for a solution when the objective function is concave.

This means that debt ($-b_t$) cannot be too big. This borrowing constraint rules out Ponzi-schemes and if $\widetilde{b}$ is a large enough (negative) number then this constraint is unlikely to be binding. The optimization problem is now given by

$$\begin{gathered} \max_{\{c_t,b_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \\ \text{s.t. } c_t + b_{t+1} \leq y_t + (1+r) b_t \\ b_{t+1} \geq \widetilde{b} \\ b_1 = \overline{b} \end{gathered} \tag{1.13}$$

When we make the assumption that the budget constraint is always binding and the debt ceiling is never binding then the first-order conditions would be equal to

$$\frac{\partial u(c_t)}{\partial c_t} = \frac{\partial u(c_{t+1})}{\partial c_{t+1}} \tag{1.14}$$

and

$$c_t + b_{t+1} = y_t + (1+r) b_t. \tag{1.15}$$

From 1.14 we learn that consumption is constant over time. But this still allows us for many admissable time paths. For example, suppose that $y_t = 10$ and $c_t = 9$ for all t. Even though this time path for consumption satisfies the intertemporal first-order condition, it clearly it is not optimal to always consume an amount less than the income earned. If $b_1 = 0$ then the time path for bond holdings implied by this consumption path satisfies

$$b_2 = 1, b_3 = 1 + (1+r), b_4 = 1 + (1+r) + (1+r)^2, \cdots . \tag{1.16}$$

Since this time path doesn't satisfy the transversality condition

$$\lim_{T \to \infty} \beta^{T-1} \frac{\partial u(c_T)}{\partial c_T} b_{T+1} = 0 \tag{1.17}$$

we know that this is not an optimal solution.

Occasionally one can hear the comment that the transversality can be used to rule out Ponzi schemes. First, note that the transversality condition is meant to do the opposite, that is, it is meant to prevent savings from becoming too large, not from becoming too negative. Although the transversality condition clearly rules out some Ponzi schemes, it cannot be used to rule out Ponzi schemes in general as becomes clear in the following example.

Suppose that utility is bounded and that, in particular, the marginal utility of consumption is equal to zero when $c_t = 5,000,000$. If the only constraints were the budget constraint and the transversality condition you could still play a Ponzi scheme, for example by choosing a consumption path such that consumption reaches a value above 5,000,000 in the limit.

1.2 Deriving first-order conditions: The uncertainty case

Now suppose that productivity also depends on a random technology shock. In particular, let output be given by

$$f(\theta_t, k_t), \tag{1.18}$$

where θ_t is a first-order Markov process. We say that θ_t is an n^{th}-order Markov process if the distribution of θ_t conditional on n lags is the same as the density of θ_t conditional on all lags. Thus,

$$prob(\theta_t < \theta | \theta_{t-1}, \cdots, \theta_{t-n}) = prob(\theta_t < \theta | \theta_{t-1}, \cdots). \tag{1.19}$$

Typical production function and law of motion for θ_t used in the literature are

$$f(\theta_t, k_t) = e^{\theta_t} k_t^{\alpha} \tag{1.20}$$

and

$$\theta_t = \rho\theta_{t-1} + \varepsilon_t, \tag{1.21}$$

where ε_t is a white noise random variable. The maximization problem can now be written as follows:

$$\begin{array}{c} \max_{\{c_t, k_{t+1}\}_{t=1}^{\infty}} \mathrm{E}\left[\sum_{t=1}^{\infty} \beta^{t-1} u(c_t) | I_0\right] \\ \text{s.t. } c_t + k_{t+1} \leq f(\theta_t, k_t) + (1-\delta)k_t \\ k_{t+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{1.22}$$

We assume that the information set at period t consists of all current and lagged values of θ_t, k_{t+1}, and c_t. Being clear about the information set and what the solution can depend on is important. The agent would like to make his period t decisions depend on future values of θ_t, but we will not allow him to do this, since these values are not yet known in period t. So a solution is a *plan* that chooses capital and consumption conditional on the realized values of θ_t.

Even if you are only interested in knowing what the time path for consumption is for one particular realized time path of θ_t, solving for this consumption series requires solving for the complete plan, that is, the consumption path for all possible realizations. The reason is that you as researcher may only be interested in studying consumption for one particular time path of θ_t, the agent faces an uncertain future and his decisions today cannot be determined without understanding what he will do in the future for all possible outcomes.

Solving the optimization problem in 1.22 implicitly requires solving the same optimization problem starting in period two, three, etc.. It is often useful to explicitly recognize this and write 1.22 as

$$\begin{gathered}\max_{\{c_{t+j},k_{t+1+j}\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty}\beta^j u(c_{t+j})|I_t\right]\\ \text{s.t. } c_{t+j}+k_{t+1+j}\le f(\theta_{t+j},k_{t+j})+(1-\delta)k_{t+j}\\ k_{t+1+j}\ge 0\\ k_t \text{ predetermined}\end{gathered}\tag{1.23}$$

In this problem, the constraints have to hold for every $j = 0, 1, 2, \cdots$. The transversality and first-order conditions for this optimization problem are equal to

$$\begin{gathered}\frac{\partial L(\cdot)}{\partial c_t}=0:\frac{\partial u(c_t)}{\partial c_t}=\lambda_t\\ \frac{\partial L(\cdot)}{\partial k_{t+1}}=0:-\lambda_t+\beta\mathrm{E}\left\{\lambda_{t+1}\left[\frac{\partial f(\theta_{t+1},k_{t+1})}{\partial k_{t+1}}+(1-\delta)\right]\right\}=0\\ \frac{\partial L(\cdot)}{\partial\lambda_t}\lambda_t=0:\left[f(\theta_t,k_t)+(1-\delta)k_t-c_t-k_{t+1}\right]\lambda_t=0\\ \frac{\partial L(\cdot)}{\partial\lambda_t}\ge 0:\left[f(\theta_t,k_t)+(1-\delta)k_t-c_t-k_{t+1}\right]\ge 0\\ \lambda_t\ge 0\\ \lim_{J\to\infty}\beta^J\mathrm{E}\left[\frac{\partial u(c_{t+J})}{\partial c_{t+J}}k_{t+J}|\,I_t\right]=0\end{gathered}\tag{1.24}$$

To construct these first-order conditions we used that $\mathrm{E}[h(x_t)|\,I_t] = h(x_t)$ when x_t is an element of I_t and $h(\cdot)$ is a measurable function.

Information set and state variables.

Above we have been somewhat vague about the information set. In principle it could include current and lagged values, but the question arises whether all this information is really useful. Let's spend some time thinking what the economic agent really needs to know in period t to make his decision. The agent clearly cares about his resources in the current period, which are determined by θ_t and k_t. He also cares about future values of the productivity shock. Since θ_t is a first-order Markov process, no information other than θ_t is useful in making predictions. It thus makes sense to assume that the agent bases his decisions on θ_t and k_t. These variables are called the state variables.[5] Suppose that θ_t is an i.i.d. random variable. This means that θ_t is not useful in predicting future values of θ_t. in this case the value of current production $y_t = \theta_t\, f(k_t)$ is a sufficient state variable and one would not need to know both θ_t and k_t. Figuring out what the state

[5] In the optimal control literature one often doesn't include the exogenous variables that affect the solution in the set of state variables. Nevertheless, it is useful to be aware of *all* variables that determine the decisions in period t especially when one plans on actually solving the system. I recommend to follow the standard practice in macro and to include in the set of state variables *all* variables that affect the agents' decisions.

variables are is not always an easy problem. In fact, there isn't always an unique choice for the set of choice variables.[6] At this point, it is typically better not to spend too much energy figuring out whether one can reduce the set of state variables. Note that in the i.i.d. case it is not wrong to use both θ_t and k_t as state variables. It is just not the most efficient choice. In contrast, if θ_t is not i.i.d. then using $\theta_t\, f(k_t)$ as the only state variable is wrong.

Beginning-of-period or End-of-Period stock variables

There is one tedious detail that we have to discuss. Above, k_t denoted beginning-of-period capital. In the literature one also encounters a different notation, namely one in which k_t stands for end-of-period capital. This obviously doesn't change the model. The budget constraint would then be equal to

$$c_t + k_t = f(\theta_t, k_{t-1}) + (1-\delta)k_{t-1} \tag{1.25}$$

and the Euler equation would be given by

$$-\frac{\partial u(c_t)}{\partial c_t} + \beta \mathrm{E}\left[\frac{\partial u(c_{t+1})}{\partial c_{t+1}}\left(\frac{\partial f(\theta_{t+1}, k_t)}{\partial k_t} + (1-\delta)\right) \mid I_t\right] = 0$$

The advantage of using end-of-period capital stock is that all variables in the information set of period t have a subscript t. But using beginning-of-period capital is less cumbersome in section 1.5 when we discuss the competitive equilibrium corresponding to this model.

1.3 Dynamic Programming

The problem described above has what is called a recursive structure. Every period the exact same problem is solved. Of course, the information set is different every period because the values of θ_t and k_t change over time. But when values of θ_t and k_t in period $t = 341$ are identical to those in period $t = 1253$ then the economic agent will make the exact same choices in both periods. Thus, we would have $k_{342} = k_{1254}$. An example of a problem that is not recursive is the finite-life problem 1.7. When $t = T - 1$ in this problem then the agent is close to the end of his life and typically will make a different investment choice than when $t = 1$, even when the state variables are the same.

We will refer to the maximized value of the optimization problem in equation 1.23 as $v(\cdot)$. Since the state variables are θ_t and k_t we know that

[6] For example, sun spot solutions may be possible in which solutions depend on variables just because agents believe these variables matter.

the value function depends on θ_t and k_t. Thus,

$$\begin{array}{c} v(\theta_t, k_t) = \max_{\{c_{t+j}, k_{t+j+1}\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \middle| I_t\right] \\ \text{s.t. } c_{t+j} + k_{t+1+j} \leq f(\theta_{t+j}, k_{t+j}) + (1-\delta)k_{t+j} \\ k_{t+1+j} \geq 0 \end{array} \tag{1.26}$$

Solving the problem starting in period t also implies behaving optimally in period $t+1$. So if one has found a function $v(\theta_t, k_t)$ that solves 1.26 then this solution also satisfies

$$\begin{array}{c} v(\theta_t, k_t) = \max_{c_t, k_{t+1}} u(c_t) + \beta \mathrm{E}\left[v(\theta_{t+1}, k_{t+1}) | I_t\right] \\ \text{s.t. } c_t + k_{t+1} \leq f(\theta_t, k_t) + (1-\delta)k_t \\ k_{t+1} \geq 0 \end{array} \tag{1.27}$$

The question arises whether the opposite is also true. That is, does a function $v(\theta_t, k_t)$ that solves 1.27 also solve 1.26. Stockey and Lucas (1989) show that under fairly weak assumptions the answer is yes if the following transversality condition is satisfied.

$$\lim_{t \to \infty} \mathrm{E}_t\left[\beta^t v(\theta_t, k_t)\right] = 0 \quad \text{for all feasible plans for } k_t. \tag{1.28}$$

The reason that you need Equation 1.28 is fairly intuitive. If you iterate on Equation 1.27 then you get an infinite sum as in 1.23 *and* $\beta^t v(\theta_t, k_t)$. Equation 1.28 ensures that this additional term goes to zero.

The equation in 1.27 is called the Bellman equation. The advantage is that it turns an infinite-horizon optimization problem in a finite-horizon optimization problem. You might think that the optimization in 1.27 is impossible to solve, since the solution, i.e., $v(\theta_t, k_t)$, shows up on the right-hand side of the equation as part of the function to be maximized. Nevertheless, working with the Bellman equation is typically easier than working with the original infinite horizon problem. Especially proving properties like existence, uniqueness, continuity, and differentiability of the solution are easier with 1.27 than with 1.26.

Deriving first-order conditions using the Bellman equation.

In the recursive formulation of the problem, we don't care anymore about the particular time period for which we are solving the problem. For this particular problem it is enough to distinguish between the current value and next period's value. Therefore, we let x denote the current-period value of the variable x and we let x_{+1} denote next period's value. When the marginal utility of consumption is positive then the budget constraint will always be binding. This allows us to substitute out consumption using the budget constraint. After doing this the problem is equal to[7]

[7] To simplify the notation we suppress the information set of the expectation in the Bellman equation. Also, we have ignored the non-negativity constraint on capital which would be binding only for irregular prefences.

$$v(\theta,k) = \max_{k_{+1}} u\left(f(\theta,k) + (1-\delta)k - k_{+1}\right) \\ +\beta E\left[v(\theta_{+1},k_{+1})\right]. \tag{1.29}$$

The first-order condition for this problem is given by

$$\frac{\partial u(\cdot)}{\partial c} = \beta \mathrm{E}\left[\frac{\partial v(\theta_{+1},k_{+1})}{\partial k_{+1}}\right] \tag{1.30}$$

The problem is that this expression contains the unknown function $v(\cdot)$. Using the envelope condition, however, we can figure out what this derivative is equal to. Differentiating 1.29 with respect to k gives

$$\frac{\partial v(\theta,k)}{\partial k} = \frac{\partial u(\cdot)}{\partial c}\left[\frac{\partial f(\theta,k)}{\partial k} + (1-\delta)\right] \tag{1.31}$$

Leading this equation one period and substituting it into 1.30 gives

$$\frac{\partial u(c)}{\partial c} = \beta \mathrm{E}\left\{\frac{\partial u(c_{+1})}{\partial c_{+1}}\left[\frac{\partial f(\theta_{+1},k_{+1})}{\partial k_{+1}} + (1-\delta)\right]\right\} \tag{1.32}$$

Note that this is the same Euler equation given above in 1.24. When we don't substitute out the budget constraint we can derive the same Euler equation. In that case the optimization problem is equal to

$$v(\theta,k) = \max_{c,k_{+1}} \min_{\lambda}\ u(c) + \beta \mathrm{E} v(\theta_{+1},k_{+1}) \\ +\lambda\left[f(\theta,k) + (1-\delta)k - c - k_{+1}\right] \tag{1.33a}$$

$$\text{s.t. } \lambda_t \geq 0 \tag{1.33b}$$

The first-order conditions for this problem are given by

$$\frac{\partial u(c)}{\partial c} = \lambda \tag{1.34a}$$

$$\lambda = \beta \mathrm{E}\left[\frac{\partial v(\theta_{+1},k_{+1})}{\partial k_{+1}}\right] \tag{1.34b}$$

$$\left[f(\theta,k) + (1-\delta)k - c - k_{+1}\right]\lambda = 0 \tag{1.34c}$$

$$\left[f(\theta,k) + (1-\delta)k - c - k_{+1}\right] \geq 0 \tag{1.34d}$$

$$\lambda \geq 0 \tag{1.34e}$$

Substituting out λ gives the key condition

$$\frac{\partial u(c)}{\partial c} = \beta \mathrm{E}\left[\frac{\partial v(\theta_{+1},k_{+1})}{\partial k_{+1}}\right] \tag{1.35}$$

Using the envelope condition, we can figure out what the derivative on the right-hand side of the equation is equal to

$$\frac{\partial v(\theta,k)}{\partial k} = \lambda\left[\frac{\partial f(\theta,k)}{\partial k} + (1-\delta)\right] \tag{1.36}$$

Leading this equation one period and substituting it into 1.35 gives 1.32.

Transversality condition

When we derive the first-order condition using dynamic programming, i.e. Equation 1.27, then there is no need for a transversality condition. The reason is that the optimiziation problem of the Bellman equation is a finite problem. Of course, the original problem is the infinite horizon problem and one can only use the Belmann if the tranversality condition 1.28 is satisfied. Thus, if one derives the first-order conditions with the Bellman equation one has to check transversality condition 1.28 and if one derives the first-order conditions directly using the Lagrangian for the infinite horizon problem, then one needs the transversality condition 1.6. We will now show that **??** under some weak conditions actually implies the transversality condition of the original problem 1.6. To simplify the analysis we focus on the case without uncertainty.[8]

If we substitute out consumption using the budget constraint, then we can write the infinite-horizon optimization problem as

$$\begin{array}{c} \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \widetilde{u}(k_t, k_{t+1}) \\ \text{s.t. } k_{t+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{1.37}$$

and the transversality condition 1.6 as

$$\lim_{T \to \infty} -\beta^{T-1} \widetilde{u}_2(k_T, k_{T+1}) k_{T+1} = 0. \tag{1.38}$$

Here $\widetilde{u}(k_t, k_{t+1})$ is defined as $u(f(k_t) + (1-\delta)k_t - k_{t+1})$. The Bellman equation can now be written as

$$v(k_t) = \max_{k_{t+1}} \widetilde{u}(k_t, k_{t+1}) + \beta v(k_{t+1}) \tag{1.39}$$

$$\text{s.t. } k_{t+1} \geq 0 \tag{1.40}$$

and if the utility function satisfies the Inada conditions, we have an interior solution and the first-order condition is equal to

$$-\widetilde{u}_2(k_t, k_{t+1}) = \beta v'(k_{t+1}) \tag{1.41}$$

The function $v(k_t)$ inherits the concavity of $\widetilde{u}(k_t, k_{t+1})$ under fairly weak regularity conditions.[9] Moreover, we assume that $\widetilde{u}(0,0) = 0$.[10] then we have

[8] The discussion here is based on Kamihigashi (2005).

[9] See Theorem 4.8 in Stockey and Lucas (1989).

[10] This is an innocuous assumption if utility is finite for all feasible time paths.

$$0 \leq -\widetilde{u}_2(k_t, k_{t+1})k_{t+1} = \beta v'(k_{t+1})k_{t+1} \leq \beta v(k_{t+1}) \tag{1.42}$$

and

$$0 \leq -\beta^{T-1}\widetilde{u}_2(k_T, k_{T+1})k_{T+1} \leq \beta^T v(k_{T+1}). \tag{1.43}$$

If the transversality condition 1.28 is satisfied, then the rightmost side converges to zero. Consequently, the middle term converges to zero as well. Thus, 1.28 implies 1.38.

Necessity of transversality condition

There are several articles in the literature that discuss whether the transversality condition is necessary for an optimal solution. We know that the transversality condition is necessary in "regular" models. How large this class of regular models is, is an interesting research topic. An important contribution is Kamihigashi (2005) who shows that the transversality condition is necessary in stochastic models with bounded or constant relative risk aversion utility. His setup covers a lot of models in macroeconomics. Note that Kamihigashi (2005) does not show that outside the class of functions considered the transversality condition is not necessary. The set of models for which we know the transversality to be necessary has increased over time and is likely to grow even further. That set of models is not, however, the complete universe of models. There are some examples where we know that the transversality condition is violated at the optimal solution. A key aspect of such known examples is that utility is not bounded. One such example is the following

$$\begin{array}{c} \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \\ \text{s.t. } k_{t+1} \geq 0 \\ k_1 = \overline{k} \end{array} \tag{1.44}$$

with $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$, $f(k_t)$ a regular neoclassical production function, *and* $\beta = 1$. The optimal time path converges to the Golden rule capital stock, i.e., the capital stock that satisfies

$$f'(k) = 1. \tag{1.45}$$

But if $\beta = 1$ and the capital stock converges to a constant, then the transversality condition is not satisfied although such a time path for capital is optimal.

1.4 Analytical Solutions to Some Special Models

We know that a solution to the optimization problem has to satisfy the first-order conditions and under regularity conditions like a concave utility and production function, the first-order conditions are typically not only necessary but also sufficient. If the problem is recursive, then knowledge about the state variables gives us a list of potential arguments for the policy functions. To solve for the actual functional form, however, one typically needs to use numerical techniques. Nevertheless, there are a couple of examples where we know how to solve for the model analytically. The best known example is probably the following version of the neoclassical growth model:

$$\begin{gathered}\max_{\{c_{t+j},h_{t+j},k_{t+1+j}\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty} \beta^j \left(\ln(c_{t+j}) + B\ln(1-h_{t+j})\right)|I_t\right]\\ \text{s.t. } c_{t+j} + k_{t+1+j} \leq \theta_{t+j}k_{t+j}^{\alpha}h_{t+j}^{1-\alpha}\\ k_{t+1+j} \geq 0\\ k_t \text{ predetermined}\end{gathered} \tag{1.46}$$

Note that this problem is just like the one in 1.23. We have only specified a particular utility and production function and assumed that capital fully depreciates within one period.[11] The first-order condition for this problem are given by

$$\frac{1}{c_t} = \beta \mathrm{E}\left[\frac{\alpha\theta_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}}{c_{t+1}}|I_t\right] \tag{1.47a}$$

$$c_t + k_{t+1} = \theta_t k_t^{\alpha} h_t^{1-\alpha} \tag{1.47b}$$

$$\frac{1}{c_t}(1-\alpha)\theta_t k_t^{\alpha} h_t^{-\alpha} = \frac{B}{1-h_t} \tag{1.47c}$$

The solutions to this set of equations are equal to

$$h_t = h = \frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)} \tag{1.48a}$$

$$c_t = (1-\alpha\beta)\theta_t k_t^{\alpha}\left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} \tag{1.48b}$$

$$k_{t+1} = \alpha\beta\theta_t k_t^{\alpha}\left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} \tag{1.48c}$$

It is easy to check that these are indeed solutions by substituting them into 1.47. Note that the solutions (also called policy functions) are functions of the current productivity shock θ_t and the capital stock k_t. The system in 1.63 gives the solution independent of what order Markov process the technology shock θ_t is. You might be surprised by this because if θ_t is, for example, a second-order Markov process then θ_{t-1} helps to predict θ_{t+1}. So the question arises why in this particular example, the investment function doesn't depend on θ_{t-1}. The reason is that 1.47a is a very special

[11] Clearly not a very realistic assumption.

Euler equation for which the argument inside the conditional expectation actually is not a random variable at all.[12] What happens here is that a higher value of θ_{t+1} raises the marginal product of capital (which should increase investment in period t) but it also lowers the marginal utility of consumption in period $t+1$ (which reduces the investment in period t). In this case the effects exactly off set each other. The economic agent is, thus, not interested in predicting θ_{t+1}. Moreover, an increase in θ_t has a wealth effect and a substitution effect on hours that exactly offset each other.

Note that in this particular example one could use current income, $\theta_t k_t^\alpha h^{1-\alpha}$, as the (single) state variable. But as we mentioned before, it is in general better not too spend too much emphasis trying to reduce the set of state variables until you understand the model well. It typically is better to have a redundant state variable (which will play no role in the policy functions) then to miss one state variable (which will typically lead to an incorrect solution).

Why hard to find an analytical solution in general?

Consider the first-order condition of the agent's optimization problem after we have used the budget constraint to substitute out consumption.

$$\begin{aligned} &u'(f(\theta,k)+(1-\delta)k-k_{+1}) \\ =\ &\beta E\left\{u'(f(\theta_{+1},k_{+1})+(1-\delta)k_{+1}-k_{+2})\left[f'(\theta_{+1},k_{+1})+(1-\delta)\right]\right\} \end{aligned} \tag{1.49}$$

We know it makes sense that the solution for k_{+1} is a function of θ and k, thus, $k_{+1}=g(\theta,k)$. Leading this expression one period we get $k_{+2}=g(\theta_{+1},k_{+1})=g(\theta_{+1},g(\theta,k))$. Substituting this function into 1.49 gives

$$\begin{aligned} &u'(f(\theta,k)+(1-\delta)k-g(\theta,k)) \\ =\ &\beta E\left\{\begin{array}{c} u'\left[f(\theta_{+1},g(\theta,k))+(1-\delta)g(\theta,k)-g(\theta_{+1},g(\theta,k))\right] \\ \times\left[f'(\theta_{+1},g(\theta,k))+(1-\delta)\right] \end{array}\right\} \end{aligned} \tag{1.50}$$

Let's make life a bit easier and use explicit functional forms and let's assume that we know the values of the parameters used. For example, let

$$u(c)=\frac{c^{1-\gamma}-1}{1-\gamma}$$

and let

$$f(\theta,k)=\theta k^\alpha.$$

Then 1.50 can be written as

[12] You can easily check this by using the policy function to substitute out c_{t+1}.

$$
\begin{aligned}
&[\theta k^{\alpha} + (1-\delta)k - g(\theta, k)]^{-\gamma} \\
&= \beta E \left\{ \begin{array}{c} [\theta_{+1} g(\theta, k)^{\alpha} + (1-\delta) g(\theta, k) - g(\theta_{+1}, g(\theta, k))]^{-\gamma} \\ \times \left[\alpha \theta_{+1} g(\theta, k)^{\alpha - 1} + (1-\delta)\right] \end{array} \right\}
\end{aligned} \tag{1.51}
$$

The problem is that $g(\cdot)$ is a function and we have a priori no idea on what the functional form is. Thus we are searching for a solution in a very big space. Now the good thing is that we have many equations to solve for this object since 1.50 has to hold for all values of θ and k. That is, we have to solve for an infinite number of values, namely a function, but we also have an infinite number of equations. Note that if we know the functional form then the problem becomes much simpler. For example, suppose the policy function is known to be linear in θ and k. Then we only need three combinations of θ and k and Equation 1.51 evaluated at these three observations gives us three equations to solve for the three coefficients of the linear policy rule. Of course, one still would have to figure out how to deal with the conditional expectation, i.e., the integration but relative to the infinite dimension of the original problem, this is actually less of a problem.

1.5 Representative Agent Assumption

In this section we will motivate the model used in sections 1.2 and 1.3. There we basically assumed that the economic agent had a little production plant in his backyard and both the consumption and the production decision are made by the same economic agent as if he is the only one in the world. The solution that comes out of this version of the model is called the social planner's solution. In a social planner's problem there are no market prices. The social planner simply maximizes the agents' utility subject to what is feasible.

You might think that this is a very silly model since any actual macro economy has many firms and consumers. And indeed it is a silly model, but not as silly as it looks at first sight. To demonstrate the last part of this claim we proceed in three steps. In the first step, we build an economy in which there are consumers who work at firms and make investment decisions and firms that hire workers and capital. In this economy there will be a wage rate and a rental price for capital and we will refer to the solution of this problem as the competitive equilibrium. In the second step, we show that the solution to this competitive equilibrium is identical to the social planner's version of the economy. In both the competitive model and the social planner's model, we work with a large number of identical agents, or equivalently with a *representative agent*. In the third step, we describe an environment in which an economy with many different agents can be described exactly with a model with one such representative agent.

1.5.1 Social Planner's Problem

We will start by extending the model by including a variable labor supply. In particular, we now assume that production also depends on the amount of labor supplied, h_t, and the current-period utility also depends on leisure, $l_t = 1 - h_t$. Using some standard functional forms the optimization problem can now be written as

$$\begin{array}{c} \max_{\{c_{t+j},h_{t+j},k_{t+1+j}\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, 1-h_{t+j})|I_t\right] \\ \text{s.t. } c_{t+j} + k_{t+1+j} \leq \theta_{t+j} k_{t+j}^{\alpha} h_{t+j}^{1-\alpha} + (1-\delta)k_{t+j} \\ k_{t+1+j} \geq 0 \\ k_t \text{ predetermined} \end{array} \tag{1.52}$$

The transversality and first-order conditions for this optimization problem are equal to

$$\begin{array}{c} \frac{\partial L(\cdot)}{\partial c_t} = 0 : \frac{\partial u(c_t,1-h_t)}{\partial c_t} = \lambda_t \\ \frac{\partial L(\cdot)}{\partial h_t} = 0 : \lambda_t(1-\alpha)\theta_t \left(\frac{k_t}{h_t}\right)^{\alpha} = -\frac{\partial u(c_t,1-h_t)}{\partial h_t} = \frac{\partial u(c_t,l_t)}{\partial l_t} \\ \frac{\partial L(\cdot)}{\partial k_{t+1}} = 0 : -\lambda_t + \beta \mathrm{E}\left\{\lambda_{t+1}\left[\alpha\theta_t \left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha-1} + (1-\delta)\right]\right\} = 0 \\ \frac{\partial L(\cdot)}{\partial \lambda_t} = 0 : c_t + k_{t+1} = \theta_t k_t^{\alpha} h_t^{1-\alpha} + (1-\delta)k_t \\ \lim_{J\to\infty} \beta^j \mathrm{E}\left[\frac{\partial u(c_{t+J},h_{t+J})}{\partial c_{t+J}} k_{t+1+J}|\, I_t\right] = 0 \end{array} \tag{1.53a}$$

1.5.2 Competitive Equilibrium

First, we will describe the behavior of firms who hire workers at the wage rate, w, and rent capital at the rental rate, r. We assume that there are a large number of firms and that markets are competitive. In this model, firms' decisions don't have any intertemporal consequences. Consequently, the standard assumption that firms maximize the net-present-value of discounted future profits implies that firms maximize profits period by period. The maximization problem for firm j is then given by

$$\max_{\tilde{k}_t^j,\tilde{h}_t^j} \theta_t \left(\tilde{k}_t^j\right)^{\alpha} \left(\tilde{h}_t^j\right)^{1-a} - r_t \tilde{k}_t^j - w_t \tilde{h}_t^j$$

where $\tilde{k}_t^j$ and $\tilde{h}_t^j$ are the demand by firm j for capital and labor, respectively. We assume that all firms face the same aggregate productivity shock θ_t. The first-order conditions for this problem are as follows:

$$\alpha\theta_t \left(\frac{\tilde{k}_t^j}{\tilde{h}_t^j}\right)^{\alpha-1} = r_t \tag{1.54a}$$

$$(1-\alpha)\theta_t \left(\frac{\tilde{k}_t^j}{\tilde{h}_t^j}\right)^{\alpha} = w_t \tag{1.54b}$$

You might think that the two equations in 1.54a and 1.54b will in general give different answers for the optimal ratio of capital to hours, $\tilde{k}_t^j/\tilde{h}_t^j$. Another awkward feature about this optimization problem is that—because of constant returns to scale—you can always double profits by doubling the amount of capital and labor used. Below we will see that the equilibrium level of the rental rate and hours are such that equilibrium profits are zero and that 1.54a and 1.54b give the same capital to hours ratio.

Constant returns to scale implies that the optimal size of the firm is not determined. That is, although in equilibrium the ratio $\tilde{k}_t^j/\tilde{h}_t^j$ is determined, the actual levels of $\tilde{k}_t^j$ and $\tilde{h}_t^j$ may be different across different firms. But firm size does not change any of the aggregate properties of the model. So without loss of generality we can pretend that there is one aggregate firm with

$$\tilde{k}_t = \sum_j \tilde{k}_t^j \text{ and } \tilde{h}_t = \sum_j \tilde{h}_t^j. \tag{1.55}$$

For this aggregate firm, the capital labor ration would be equal to $\tilde{k}_t/\tilde{h}_t = \tilde{k}_t^j/\tilde{h}_t^j \; \forall j$. Thus, $\tilde{k}_t$ and $\tilde{h}_t$ are the aggregate *demand* for capital and labor. It is easy to see that 1.54a and 1.54b hold for the aggregate capital stock, $\tilde{k}_t$, and the aggregate number of hours worked, $\tilde{h}_t$. First rewrite 1.54a and 1.54b as follows:

$$\overline{k}_t^j = \left(\frac{r_t}{\alpha\theta_t}\right)^{1/(\alpha-1)} \overline{h}_t^j \tag{1.56a}$$

$$\overline{h}_t^j = \left(\frac{r_t}{(1-\alpha)\theta_t}\right)^{-1/\alpha} \overline{k}_t^j \tag{1.56b}$$

It is clear that 1.56a and 1.56b can easily be aggregated since all firms face the same prices. Thus,

$$\tilde{k}_t = \left(\frac{r_t}{\alpha\theta_t}\right)^{1/(\alpha-1)} \tilde{h}_t \tag{1.57a}$$

$$\tilde{h}_t = \left(\frac{r}{(1-\alpha)\theta}\right)^{-1/\alpha} \tilde{k}_t \tag{1.57b}$$

which implies that

$$\alpha\theta_t \left(\frac{\tilde{k}_t}{\tilde{h}_t}\right)^{\alpha-1} = r_t \tag{1.58a}$$

$$(1-\alpha)\theta_t \left(\frac{\tilde{k}_t}{\tilde{h}_t}\right)^{\alpha} = w_t \tag{1.58b}$$

We will now analyze the consumer problem. We assume that there are a large number of consumers who also take prices as given. In particular, the problem of the i^{th} consumer who can supply labor each period at rate w_t and rent capital at rate r_t is given by

$$\begin{array}{c} \max_{\{c^i_{t+j},h^i_{t+j},k^i_{t+1+j}\}_{j=0}^{\infty}} \mathrm{E}\left[\sum_{j=0}^{\infty}\beta^j u(c^i_{t+j},1-h^i_{t+j})|I_t\right] \\ \text{s.t. } c^i_{t+j}+k^i_{t+1+j}\leq r_t k^i_{t+j}+w_t h^i_{t+j}+(1-\delta)k^i_{t+j} \\ k^i_{t+1+j}\geq 0 \\ k_t \text{ predetermined} \end{array} \tag{1.59}$$

The transversality and first-order conditions for this optimization problem are equal to

$$\begin{array}{c} \frac{\partial L(\cdot)}{\partial c^i_t}=0:\frac{\partial u(c^i_t,1-h^i_t)}{\partial c^i_t}=\lambda^i_t \\ \frac{\partial L(\cdot)}{\partial h^i_t}=0:\lambda^i_t w_t=-\frac{\partial u(c^i_t,1-h^i_t)}{\partial h^i_t} \\ \frac{\partial L(\cdot)}{\partial k^i_{t+1}}=0:-\lambda^i_t+\beta\mathrm{E}\left\{\lambda^i_{t+1}\left[r_{t+1}+(1-\delta)\right]\right\}=0 \\ \frac{\partial L(\cdot)}{\partial \lambda^i_t}=0:c^i_t+k^i_{t+1}=\theta_t\left(k^i_t\right)^{\alpha}\left(h^i_t\right)^{1-\alpha}+(1-\delta)k^i_t \\ \lim_{J\to\infty}\beta^J\mathrm{E}\left[\lambda^i_{t+J}k^i_{t+1+J}|\,I_t\right]=0 \end{array} \tag{1.60a}$$

The per capita (or aggregate) choice variables corresponding to the individual choices are defined by[13]

$$\begin{array}{c} \bar{c}_t=\sum_{i=1}^{I}c^i_t/I \\ \bar{k}_{t+1}=\sum_{i=1}^{I}k^i_{t+1}/I \\ \bar{h}_t=\sum_{i=1}^{I}h^i_t/I \end{array} \tag{1.61}$$

From now on I drop the i superscript because it should be clear whether we are talking about an individual level variable or a per capita variable.

We are now ready to define a competitive equilibrium.

Definition (competitive equilibrium): *A competitive equilibrium consists of a consumption function, $c(k_t,\bar{k}_t,\theta_t)$, a labor supply function, $h(k_t,\bar{k}_t,\theta_t)$, a capital supply function, $k_{+1}(k_t,\bar{k}_t,\theta_t)$, an aggregate per capita consumption function, $\bar{c}(\bar{k}_t,\theta_t)$, aggregate per capita capital supply function, $\bar{k}_{+1}(\bar{k}_t,\theta_t)$, aggregate per capita labor supply function, $\bar{h}(\bar{k}_t,\theta_t)$, an aggregate capital demand function $\tilde{k}(\bar{k}_t,\theta_t)$,* an aggregate labor demand function $\tilde{h}(\bar{k}_t,\theta_t)$, a *wage function, $w(\bar{k}_t,\theta_t)$, and a rental rate, $r(\bar{k}_t,\theta_t)$, that*

- solve the household's optimization problem,
- solve the firm's optimization problem,

[13] If we have a unit mass of agents then the per capita values of (for example) hours is defined as $\bar{h}_t=\int_0^1 h^i di$. In this case the representative firm would have to hire an infinite number of workers (and capital) and it would make more sense to assume that there is not one but also a unit mass of firms in which case the equilibrium condition specifies that the per capita supply of capital is equal to the per capita demand of capital.

- satisfy the equilibrium conditions

$$\tilde{k}(\bar{k}_t, \theta_t) = I\bar{k}(\bar{k}_t, \theta_t) \text{ and } \tilde{h}(\bar{k}_t, \theta_t) = I\bar{h}(\bar{k}_t, \theta_t)$$

and the aggregate budget constraint:

$$\bar{c}(\bar{k}_t, \theta_t) + \bar{k}_{+1}(\bar{k}_t, \theta_t) = \theta \bar{k}_t^{\alpha} \bar{h}(\bar{k}_t, \theta_t)^{1-\alpha} + (1-\delta)\bar{k}_t.$$

- are consistent with each other, that is,
  - $\bar{c}\left(\bar{k}_t, \theta_t\right) = c(\bar{k}_t, \bar{k}_t, \theta_t)$,
  - $\bar{k}_{+1}\left(\bar{k}_t, \theta_t\right) = k_{+1}(\bar{k}_t, \bar{k}_t, \theta_t)$, and
  - $\bar{h}\left(\bar{k}_t, \theta_t\right) = h(\bar{k}_t, \bar{k}_t, \theta_t) \, \forall \bar{k}_t, \forall \theta_t$

Consistency

The last requirement is probably the hardest to understand. It implies the aggregation condition 1.61 *for our representative agent framework,* but is in general a bit weaker. We will use this framework to study the case in which everybody is identical.[14] If everybody is identical then it must be true that for each agent the individual capital stock, k_t^i, is equal to $\bar{k}_t$ and everybody will choose $c(\bar{k}_t, \bar{k}_t, \theta_t)$. Using the explicit definition of $\bar{c}_t$ given in 1.61, we get that $\bar{c}_t = c(\bar{k}_t, \bar{k}_t, \theta_t)$, which is exactly the consistency requirement. The beauty of this more general setup is that—even though everybody is the same—we can still answer the question how one individual agent's decision is going to change if we change his individual capital stock but leave per capita capital the same.[15]

Note that both the individual capital stock, k, and the per capita, $\bar{k}$, are arguments of the agent's individual decision rules. Why would an agent care about the aggregate capital stock? Let's think what an individual cares about in period t. Clearly he cares about k_t as well as current and future wages and rental rates. It seems there is no reason why he would care about the capital stock of other agents in the economy.

So the question arises whether it would make sense to make wages and rental rates arguments of the policy functions? The first problem you will encounter is that they are not predetermined variables. The second (related) problem is that we wouldn't know how many lags of wages and rental rates to include since we don't know what order Markov process these prices are.[16] Now let's think whether there are predetermined variables that affect current wages and rental rates and/or have predictive

[14] Although agents are of the same type they could still be different because of initial conditions, that is, different initial capital stocks but those differences would gradually disappear over time.

[15] That is, as long as their mass is zero I can make some agents different from the rest of the economy without changing anything.

[16] The productivity process being a first-order Markov process does in general not imply that endogenous variable in the model are first-order Markov processes.

power for future values of these prices. Those are the current productivity level θ and the capital stocks of the other agents. But since everybody is the same in this economy the rest of the economy is described completely by the per capita (or the aggregate) capital stock.[17]

But if everybody is the same, then we also know that the capital stock of the agent we are considering, k, must be equal to the per capita capital stock, $\bar{k}$. And in fact we often exploit this property. Nevertheless, it is important to realize that you cannot impose on the individual problem that k_t will always be equal to $\bar{k}_t$. In equilibrium, prices are such that the individual, who has the freedom to do something different than the other agents in the economy, does exactly what everybody else does. Moreover, the researcher may want to know how one individual's behavior changes if his individual capital stock increases but the aggregate capital stock (that is the capital holdings of the other agents in the economy) does not.

By comparing the first-order conditions of the social planner's problem with those of the competitive equilibrium it is clear that there is a choice for the wage and the rental rate such the equations for the competitive equilibrium coincide with those of the social planner. In particular, let the wage rate be equal to the marginal product of labor and let the rental rate be equal to the marginal product of capital. In this model, the social planner's problem can, thus, be obtained in a competitive equilibrium. There are many models for which the allocations of the social planner's solution are not equal to those obtained in the competitive equilibrium or only under special circumstances. In the next chapter we will encounter such examples.

Analytical example

Above we showed that if agents had log utility (for consumption and leisure) and if capital depreciates fully (a not so realistic assumption) that there is an analytical solution to the social planner's version of the model. There is also an analytical solution to the competitive equilibrium. It is given below. It is a good exercise to check yourself.

Individual policy rules

$$h(k_t, \bar{k}_t, \theta_t) = h = \frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)} \tag{1.62a}$$

$$c(k_t, \bar{k}_t, \theta_t) = (1-\alpha\beta)\theta_t \bar{k}_t^{\alpha-1}\left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} k_t \tag{1.62b}$$

$$k_{+1}(k_t, \bar{k}_t, \theta_t) = \alpha\beta\theta_t \bar{k}_t^{\alpha-1}\left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} k_t \tag{1.62c}$$

[17] If agents start out with different capital stocks, the agents' capital stocks will converge since in this economy the initial condition only has a temporary effect.

Aggregate policy rules

$$\bar{h}(\bar{k}_t, \theta_t) = h = \frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)} \tag{1.63a}$$

$$\bar{c}(\bar{k}_t, \theta_t) = (1-\alpha\beta)\theta_t \bar{k}_t^{\alpha} \left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} \tag{1.63b}$$

$$\bar{k}_{+1}(\bar{k}_t, \theta_t) = \alpha\beta\theta_t \bar{k}_t^{\alpha} \left(\frac{1-\alpha}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} \tag{1.63c}$$

$$\tilde{k}_t = I\bar{k}_t \tag{1.64}$$

$$\tilde{h}_t = \frac{I(1-\alpha)}{B(1-\alpha\beta)+(1-\alpha)} \tag{1.65}$$

Prices

$$r_t = \alpha\theta_t \left(I\bar{k}_t\right)^{\alpha-1} \left(\frac{I(1-\alpha)}{B(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha} \tag{1.66a}$$

$$w_t = (1-\alpha)\theta_t \left(I\bar{k}_t\right)^{\alpha} \left(\frac{I(1-\alpha)}{B(1-\alpha\beta)+(1-\alpha)}\right)^{-\alpha} \tag{1.66b}$$

1.5.3 Complete Markets and the Representative Agent Assumption

The third step of what we try to accomplish in this section is to show that working with a representative consumer might not be as silly as it looks at first. To simplify the discussion, we focus on an endowment economy where all agents have identical power utility functions and face an stochastic process that generates random draws of the individual endowment shock y_t^i. These laws of motions can differ across agents. Although y_t^i could have a common component we assume that it also has an important idiosyncratic component. Therefore, even if all agents face the same law of motion, realizations of shocks will differ across agents. We will assume that the financial markets are complete. To explain the idea behind complete markets assume that given the information available in period t there are J states of nature that can occur in period $t+1$.[18] A contingent claim is an asset that delivers one unit in state $j_{t+1} \in J$. Let the price of this asset be q_t^j. Asset markets are said to be complete if all J contingent claims can be traded. In that case you can insure yourself against any random

[18] To simplify the notation, it is assumed that the number of states do not depend on t.

event. Let j^* be the current state. The recursive formulation of agents i^{th} maximization problem is given by

$$\max_{c^i, b_{+1}^{1,i}, \cdots, b_{+1}^{J,i}} \frac{\left(c^i\right)^{1-\gamma}}{1-\gamma} + \beta \mathrm{E} v(b_{+1}^{1,i}, \cdots, b_{+1}^{J,i})$$
$$s.t. c^i + \sum_{j=1}^{J} q^j b_{+1}^{j,i} = y^i + \sum_{j=1}^{J} I(j^*) b^{j,i}$$
$$b_{+1}^{j,i} > \overline{b} < 0$$

where $b_{+1}^{j,i}$ is the amount of contingent claims for state j that agent i has bought and $I(j^*)$ is an indicator function that is equal to 1 if $j = j^*$ and 0 otherwise. Contingent claims are in zero net supply so that the sum of $b_{+1}^{j,i}$ across agents is equal to zero. Note that we have assumed a short-sale constraint on agent to prevent Ponzi schemes. If we assume that this constraint is not binding then, the first-order conditions for this problem are the budget constraint and the following Euler equations:

$$q^j \left(c^i\right)^{-\gamma} = \beta \left(c_{+1}^{j,i}\right)^{-\gamma} \mathrm{prob}(j)\ \forall j \tag{1.67}$$

This can be written as follows:

$$c^i = \left(\frac{\beta \mathrm{prob}(j)}{q^j}\right)^{-1/\gamma} c_{+1}^{j,i}\ \forall j \quad \text{or} \tag{1.68}$$
$$C = \left(\frac{\beta \mathrm{prob}(j)}{q^j}\right)^{-1/\gamma} C_{+1}^{j}\ \forall j, \tag{1.69}$$

where C is aggregate consumption in the current period and C_{+1}^j is aggregate consumption next period when state j occurs. Note that 1.69 can be written as

$$q^j \left(C\right)^{-\gamma} = \beta \left(C_{+1}^{j}\right)^{-\gamma} \mathrm{prob}(j)\ \forall j \tag{1.70}$$

When we use the equilibrium condition that contingent claims are in zero net supply, we get that aggregate consumption equals aggregate income and

$$q^j \left(Y\right)^{-\gamma} = \beta \left(Y_{+1}^{j}\right)^{-\gamma} \mathrm{prob}(j)\ \forall j \tag{1.71}$$

But these also would be the equilibrium equations for the maximization problem for the following representative agent

$$\max_{C,B_{+1}^{1},\cdots,B_{+1}^{J}} \frac{(C)^{1-\gamma}}{1-\gamma} + \beta \mathrm{E} v(B_{+1}^{1},\cdots,B_{+1}^{J})$$

$$s.t. C^{i} + \sum_{j=1}^{J} q^{j} B_{+1}^{j} = Y + \sum_{j=1}^{J} I(j^{*}) B^{j}$$

$$B_{+1}^{j} > \overline{b} < 0$$

You, thus, can get the same asset prices with the representative agent economy as with the economy with heterogeneous agents.

That aggregation is possible is surprising to many economists, especially econometricians. For example, we know that aggregation of two AR(1) processes does not give you another AR(1) process unless the autoregressive coefficient is the same for the two processes. But in this example, aggregate consumption is not just the sum of some exogenously specified consumption processes. Key is that all agents adjust their marginal rates of substitution such that they equal market prices and since all agents face the same prices they will have the same marginal rate of substitution.

One misperception of representative agent models is that they are unrealistic because there is no trade in equilibrium. This is of course a ridiculous statement because the representative agent model is obtained by assuming that trade in all contingent assets takes place. For example, this assumes that agents can insure against idiosyncratic shocks. A more sensible criticism would be that representative agent models assume that there is too much trade since in reality many types of contingent claims are not being traded because of moral hazard or adverse selection problems.

1.6 Growth in the standard neoclassical model

The models discussed so far assumed that the productivity process was stationary. Consequently, the generated series such as output and consumption are stationary too. The observed series, however, are growing. The growth literature tries to answer the question why there is growth. Here we address much simpler questions. The first question is whether we can impose restrictions on the model so that the growth properties of the endogenous variables correspond to what we observe. This question we answer using a steady state version of the model. The second question is whether we can transform the model so that we can analyze stationary deviations from a growth path.

1.6.1 Properties of balanced growth

We say that the model satisfies balanced growth if

1. There are no trends in the capital and labor shares, that is, in $r_t k_t / y_t$ and $w_t h_t / y_t$,
2. The rental rate has no trend and the wage rate grows at the same rate as output, and
3. y_t / h_t and k_t / h_t grow at a roughly constant rate.

We will now show how the neoclassical growth model can generate these facts. An essential assumption is that there is labor augmenting growth. That is, the production function is written as

$$y_t = \widetilde{f}_t(k_t, h_t) = f(k_t, (1+\gamma)^t h_t), \tag{1.72}$$

where (exogenous) technological growth is captured by $(1+\gamma)^t$. Note that $\widetilde{f}$ does and f does not have a time subscript. Note that this assumption is clearly satisfied if the production function is given by

$$y_t = k_t^{\alpha} \left((1+\gamma)^t h_t \right)^{1-\alpha}. \tag{1.73}$$

We focus on solutions that are of the following form

$$x_t = x_0 (1+g_x)^t \tag{1.74}$$

for x_t equal to y_t, c_t, k_t, i_t, and h_t. Now we will check whether the model imposes restrictions on the growth rates of the variables. From

$$k_{t+1} = (1-\delta) k_t + i_t \tag{1.75}$$

we get that $g_k = g_i$. From

$$y_t = c_t + i_t \tag{1.76}$$

we get that $g_y = g_c = g_i$.[19] Constant returns to scale gives

$$y_t = f(k_t, (1+\gamma)^t h_t) = (1+\gamma)^t h_t f\left(\frac{k_t}{(1+\gamma)^t h_t}, 1 \right) \tag{1.77}$$

or

$$\frac{y_t}{(1+\gamma)^t h_t} = f\left(\frac{k_t}{(1+\gamma)^t h_t}, 1 \right). \tag{1.78}$$

[19] Note that we are looking for a solution in which growth rates are constant. That is, if x_t, y_t, and z_t have constant growth rates and $z_t = x_t + y_t$ then it must be the case that the growth rates are equal to each other.

We know that $y_t/((1+\gamma)^t h_t)$ and $k_t/((1+\gamma)^t h_t)$ have equal growth rates. But this is only consistent with diminishing returns of $f(\cdot, 1)$ if that growth rate is equal to zero. Thus, $g_y = g_k = \gamma + g_h$.

Now suppose that the utility function can be written as

$$u(c_t, 1-h_t) = \frac{c_t^{1-\eta} v(1-h_t) - 1}{1-\eta}. \tag{1.79}$$

The first-order condition for leisure is given by

$$\frac{c_t^{1-\eta}}{1-\eta} \frac{\partial v(1-h_t)}{\partial h_t} + c_t^{-\eta} v(1-h_t)(1+\gamma)^t f_2(k_t, (1+\gamma)^t h_t) = 0. \tag{1.80}$$

If we assume that the production function is homogeneous of degree 1 then the partial derivatives are homogenous of degree 0, thus

$$\frac{1}{1-\eta} \frac{\partial v(1-h_t)}{\partial h_t} + c_t^{-1} v(1-h_t)(1+\gamma)^t f_2\left(\frac{k_t}{(1+\gamma)^t}, 1\right) = 0. \tag{1.81}$$

From the discussion above we know that $k_t/(1+\gamma)^t$ is constant. Thus, this equation is consistent with $g_c = \gamma$ and $g_h = 0$. Combining this with the results obtained above we get that $g_y = g_i = g_k = \gamma$. The marginal product of capital

$$r_t = f_1(k_t, (1+\gamma)^t h_t) = f_1\left(\frac{k_t}{(1+\gamma)^t}, 1\right) \tag{1.82}$$

is constant so that the capital share $r_t k_t / y_t$ is constant as well. The marginal product of labor

$$w_t = (1+\gamma)^t f_2\left(\frac{k_t}{(1+\gamma)^t}, 1\right) \tag{1.83}$$

grows at rate γ so that the wage share $w_t h_t / y_t$ is constant as well.

1.6.2 Stationarity-inducing transformation

We now show how we can transform the model into one with only stationary variables. The orginal problem is given by

$$\begin{gathered}\max_{\{c_t, h_t, k_{t+1}\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\eta} v(1-h_t) - 1}{1-\eta} \\ \text{s.t.} \;\; c_t + k_{t+1} \le k_t^{\alpha} (\theta_t h_t)^{1-\alpha} + (1-\delta) k_t \\ k_{t+1} \ge 0, k_1 = \overline{k}\end{gathered}$$

We will now describe transformations of the model so that we come up with a system of equations for stationary variables for different types of processes for θ_t. In particular, θ_t could be a trend-stationary process, i.e.,

$$\theta_t = (1+g_\theta)^t, \tag{1.84}$$

and θ_t could be difference-stationary, i.e.,

$$\ln \theta_t = \ln \theta_{t-1} + u_t, \tag{1.85}$$

where u_t is a stationary process. But in fact all steps performed here are legitimate if θ_t is a stationary process.The first-order conditions for this problem are given by

$$\lambda_t = c_t^{-\eta} v(1 - h_t) \tag{1.86}$$

$$\lambda_t = \mathrm{E}_t \left[\beta \lambda_{t+1} \left(\alpha \theta_{t+1}^{1-\alpha} \left(\frac{k_{t+1}}{h_{t+1}} \right)^{\alpha - 1} + (1 - \delta) \right) \right] \tag{1.87}$$

$$\frac{c_t^{1-\eta} v'(1 - h_t)}{1 - \eta} = \lambda_t (1 - \alpha) \theta_t^{1-\alpha} k_t^{\alpha} h_t^{-\alpha} \tag{1.88}$$

The first step of the transformation is to define the following variables

$$\begin{aligned} \tilde{c}_t &= c_t / \theta_t, & (1.89)\\ \tilde{k}_t &= k_t / \theta_t, & (1.90)\\ \tilde{h}_t &= h_t, & (1.91)\\ \tilde{\lambda}_t &= \lambda_t / \theta_t^{-\eta} & (1.92) \end{aligned}$$

Using these, we can rewrite the first-order conditions and the budget constraint as

$$\tilde{\lambda}_t = \tilde{c}_t^{-\eta} v(1 - \tilde{h}_t) \tag{1.93}$$

$$\tilde{\lambda}_t = \mathrm{E}_t \left[\beta \tilde{\lambda}_{t+1} \left(\frac{\theta_{t+1}}{\theta_t} \right)^{-\eta} \left(\alpha \left(\frac{\tilde{k}_{t+1}}{h_{t+1}} \right)^{\alpha - 1} + (1 - \delta) \right) \right] \tag{1.94}$$

$$\frac{\tilde{c}_t^{1-\eta} v'(1 - h_t)}{1 - \eta} = \tilde{\lambda}_t (1 - \alpha) \tilde{k}_t^{\alpha} h_t^{-\alpha} \tag{1.95}$$

$$\widetilde{c}_t + \left(\frac{\theta_{t+1}}{\theta_t} \right) \widetilde{k}_{t+1} \leq \widetilde{k}_t^{\alpha} \widetilde{h}_t^{1-\alpha} + (1 - \delta) \widetilde{k}_t \tag{1.96}$$

Note that $\left(\frac{\theta_{t+1}}{\theta_t} \right) \widetilde{k}_{t+1}$ is not a stochastic variable.

Consider the case in which (the log of) θ_t is a random walk and the growth rate of θ_t is, thus, an i.i.d. random variable. In this case the only state variable of the model is the transformed capital stock, $\tilde{k}_t = k_t/\theta_t$. It may be tricky to understand this by looking at the equations but it is fairly intuitive. Suppose I start my economy with $k_1 = 100$ and $\theta_1 = 1$. Now, I consider an alternative economy in which $k_1 = 200$ and $\theta_1 = 2$. The second economy is simply a scaled up verson of the first. So scaled decisions should be the same in both economies and they only depend on $\tilde{k}_t$. The random walk property is important. If productivity growth is serially correlated you want to include past growth rates as a state variable.

The property that the only state variable is scaled capital also depends on the production function having constant returns to scale and growth being labor augmenting.

1.7 Exercises

Exercise 1.1: *Consider again the sequence problem given in 1.1 but suppose investments made in period t will become productive only in period t+2. That is the budget constraint is given by*

$$c_t + k_{t+1} = f(k_{t-1}) + (1 - \delta)k_{t-1}.$$

Thus in period t end of period $t+1$ capital k_{t+1} is chosen and in period t both k_{t-1} and k_t are given. Write down the first-order conditions for this problem using the Lagrangian for the sequence problem and the Bellman Equation.

Exercise 1.2: *Consider again the sequence problem given in 1.1 but suppose that the current-period utility function depends on both current period consumption, c_t, and last period's consumption, c_{t-1}. Write down the first-order conditions for this problem using the Lagrangian for the sequence problem and the Bellman Equation.*