parts of Models with Heterogeneous Agents Introduction & Different Procedures to Simulate Models with Heterogenous Agents

Wouter J. Den Haan

University of Amsterdam

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Overview

- "Simple" model with heterogeneous agents
  - understanding the complexity of these models
  - role of aggregate uncertainty
  - role of incomplete markets
- Solving the Aiyagari model
  - basic numerical techniques (refresher)
- Does heterogeneity matter?

Individual agent

- Subject to employment shocks:
  - $e_{i,t} \in \{0,1\}$
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint $k_{i,t+1} \geq 0$

Laws of motion

- z_t can take on two values
- e_t can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that
  - unemployment rate only depends on current z_t
  - thus
    - $u_t = u^b$ if $z_t = z^b$ &
    - $u_t = u^g$ if $z_t = z^g$
    - with $u^b > u^g$.

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathrm{E} \sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1-\delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

- this is a relatively simple problem
- **if** processes for r_t and w_t are given

Individual agent - foc

$$
\begin{aligned}
\frac{1}{c_{i,t}} &\geq \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] \\
0 &= k_{i,t+1} \left(\frac{1}{c_{i,t}} - \beta \mathsf{E}_t \left[\frac{1}{c_{i,t+1}} \left(r_{t+1} + 1 - \delta \right) \right] \right) \\
c_{i,t} + k_{i,t+1} &= r_t k_{i,t} + (1 - \tau_t) w_t \bar{l} e_{i,t} + \mu w_t (1 - e_{i,t}) + (1 - \delta) k_{i,t} \\
k_{i,t+1} &\geq 0
\end{aligned}
$$

What aggregate variables do agents care about?

- current **and** future values of r_t and w_t
- the current values of r_t and w_t
  - only depend on aggregate capital stock, K_t, & z_t
  - !!! This is not true in general for equilibrium prices

What aggregate variables do agents care about?

- the future values, i.e., $r_{t+\tau}$ and $w_{t+\tau}$ with $\tau > 0$ depend on
  - future values of mean capital stock, i.e. $K_{t+\tau}$, & $z_{t+\tau}$
- $\Longrightarrow$ agents are interested in all information that forecasts K_t
- $\Longrightarrow$ typically this includes the complete cross-sectional distribution of employment status and capital levels **even when** you only forecast futures means

Equilibrium - first part

- Individual policy functions that solve agent's max problem
- A wage and a rental rate given by equations above.

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - $f_t =$ cross-sectional distribution of beginning-of-period capital and the employment status *after* the employment status has been realized.
  - z_{t+1} does *not* affect the cross-sectional distribution of capital
  - z_{t+1} does affect the *joint* cross-sectional distribution of capital and employment status

Transition law & continuum of agents

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

Why is this an exact equation without additional noise?

- continuum of agents $\Longrightarrow$ rely on law of large numbers to average out idiosyncratic risk
- are we allowed to do this?

Recursive equilibrium?

Questions

1. Does an equilibrium exist?
    1. If yes, is it unique?
2. Does a recursive equilibrium exist?
    1. If yes, is it unique?
    2. If yes, what are the state variables?

Recursive equilibrium?

Jianjun Miao (JET, 2006): a recursive equilibrium exist for following state variables:

- usual set of state variables, namely
  - individual shock, $e_{i,t}$
  - individual capital holdings, $k_{i,t}$
  - aggregate productivity, z_t
  - joint distribution of income and capital holdings, f_t
- and *cross-sectional distribution of expected payoffs*

Unique?

Heterogeneity $\Longrightarrow$ more reasons to expect multiplicity

- my actions depend on what I think others will do
- heterogeneity tends to go together with frictions and multiplicity more likely with frictions
  - e.g. market externalities
- more on this later

Tough numerical problem

- Suppose that recursive RE for usual state space exist
  - $s_{i,t} = \{e_{i,t}, k_{i,t}, s_t\} = \{e_{i,t}, k_{i,t}, z_t, f_t\}$
- Equilibrium:
  - $c(s_{i,t})$
  - $k(s_{i,t})$
  - $r(s_t)$
  - $w(s_t)$
  - $\Upsilon(z_{t+1}, z_t, f_t)$

Alternative representation state space

- Suppose that recursive RE for usual state space exist
  - $s_{i,t} = \{e_{i,t}, k_{i,t}, s_t\} = \{e_{i,t}, k_{i,t}, z_t, f_t\}$
- What determines current shape f_t?
  - z_t, z_{t-1}, f_{t-1} or
  - $z_t, z_{t-1}, z_{t-2}, f_{t-2}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, f_{t-3}$ or
  - $z_t, z_{t-1}, z_{t-2}, z_{t-3}, z_{t-4}, f_{t-4}$ or
  - $\cdots$

No aggregate uncertainty

$$s_t = \lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- Why is this useful from a numerical point of view?
  - when z_t is stochastic
  - when z_t is not stochastic (case of no aggregate uncertainty)

No aggregate uncertainty

State variables

$$\lim_{n \longrightarrow \infty} \{z_t, z_{t-1}, \cdots, z_{t-n}, f_{t-n}\}$$

- **If**
  1. $z_t = z \ \forall t$ and
  2. effect of initial distribution dies out
- **then** s_t constant
  - distribution still matters!
  - but it is no longer a *time-varying* argument

Somewhat easier model

- Replace borrowing constraint by penalty function
  - going short possible but costly
- workers have productivity insteady of unemployment shocks
  - $z_{i,t}$ with $\mathrm{E}[z_{i,t}] = 1$

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} \mathsf{E}\sum_{t=0}^{\infty} \beta^t \ln(c_{i,t}) - \frac{\zeta_1}{\zeta_0}\exp(-\zeta_0 k_{i,t}) - \zeta_2 k_{i,t}$$

s.t.

$$c_{i,t} + k_{i,t} = r_t k_{i,t-1} + w_t e_{i,t} + (1-\delta)k_{i,t-1}$$

First-order condition

$$-\frac{1}{c_{i,t}} + \zeta_1 \exp(-\zeta_0 k_{i,t}) - \zeta_2 + \mathsf{E}_t\left[\frac{\beta}{c_{i,t+1}}\left(r_{t+1} + 1 - \delta\right)\right] = 0$$

Penalty function

- advantage of ζ_2 term:
  - suppose $\bar{k}$ and $\bar{r}$ are steady states of rep agent model
  - if

$$\zeta_2 = \zeta_1 \exp(-\zeta_0 \bar{k})$$

    then steady state of this model is same

Equilibrium

- Unit mass of workers, $L_t = 1$
- Competitive firm $\Longrightarrow$ agent faces competitive prices
  - $w_t = (1-\alpha) K_t^{\alpha} L_t^{1-\alpha} = (1-\alpha) K_t^{\alpha}$
  - $r_t = \alpha K_t^{\alpha-1} L_t^{\alpha} = \alpha K_t^{\alpha-1}$
- No aggregate risk so

$$K_t = K$$

- How to find the equilibrium K?

Algorithm

- Guess a value for r
- This implies values for K^{demand} and w
- Solve the individual problem with these values for r & w
- Simulate economy & calculate the supply of capital, K^{supply}
- If $K^{\text{supply}} < K^{\text{demand}}$ then r too low so raise r, say

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

- Iterate until convergence

Algorithm

Using

$$r^{\text{new}} = r + \lambda(K^{\text{demand}} - K^{\text{supply}})$$

to solve

$$K^{\text{demand}}(r) = K^{\text{supply}}(r)$$

not very efficient

- Value of λ may have to be very low
- In homework we will use a better algorithm

Use Dynare to solve indiv. policy rule

- Specify guess for r in mother Matlab file
- Make r parameter in *.mod file
- In mother Matlab file write r using

```
save r_file r
```

Use Dynare to solve indiv. policy rule

- In *.mod file use

```
load r_file
set_param_value('r',r)
```

instead of

```
r = 0.013;
```

Simulate yourself using Dynare solution

1. Use values stored by Dynare or
2. Replace Dynare's disp_dr.m with my alternative

- this saves the policy functions *exactly as shown on the screen*
    - asa matrix
    - in a Matlab data file `dynarerocks.mat`
    - under the name `decision`

Two different ways to go

- Simulate a panel with a large number of agents
  - This uses Monte Carlo integration to calculate cross-sectional moments
- Use tools from numerical literature
  - grid method that requires the inverse of the policy function
  - grid method that does not require the inverse of the policy function
  - non-grid method

What is given?

- A policy function $k'(k_{i,t}, e_{i,t}, s_t)$
  - s_t: the aggregate state variables
- A fine grid and initial distribution for $t = 1$, $\{\kappa_i, p_{i,1}^{e=0}, p_{i,1}^{e=1}\}$
  - characterizes the density of capital holdings of the employed and unemployed.

What is given? (continued)

- $p^e_{i,t}$ is the probability that $k^e_t = \kappa_i$, $i = 0$ and $\kappa_{i-1} < \kappa_i$, $i = 1, 2, \cdots$
  - distribution has no mass in between grid points
- Binding constraint $\Longrightarrow p^e_{0,t} > 0$ (and CDF has some jumps at other points)

Grid method I

- Only mass *AT* grid points
  - densities are approximated with discrete valued functions
- Calculate the end-of-period distribution as follows
  - nodes correspond to beginning-of-period distribution
  - go through the nodes, κ_i, one by one

Grid method I

- Fix employment status
  - remain within the period for now
- Nodes correspond with *beginning-of-period* distribution

Grid method I

- focus on node j with mass $p_t^{e,j}$ and capital value κ_j
- $k'(\kappa_j, e, \cdot)$ can end up where relative to node i?

$$
\begin{array}{l}
k'(\kappa_j, e, \cdot) \le \kappa_{i-1} \\
\kappa_{i-1} < k'(\kappa_j, e, \cdot) < \kappa_i \\
k'(\kappa_j, e, \cdot) = \kappa_i \\
\kappa_i < k'(\kappa_j, e, \cdot) < \kappa_{i+1} \\
k'(\kappa_j, e, \cdot) \ge \kappa_{i+1}
\end{array}
$$

Grid method I

- Set end-of-period fractions:

$$f_t^{e,i} = 0 \quad \forall i$$

- Go through all nodes and allocate beginning-of-period $p_t^{e,j}$ to end-of-period $f_t^{e,i}$:

$$f_t^{e,i} = f_t^{e,i} + p_t^{e,j} \omega_t^{i,j}$$

where

$$\omega_t^{i,j} = \begin{cases} 0 & \text{if } k'(\kappa_j, e, \cdot) \le \kappa_{i-1} \\ \frac{k'(\kappa_j,e,\cdot) - \kappa_{i-1}}{\kappa_i - \kappa_{i-1}} & \text{if } \kappa_{i-1} < k'(\kappa_j, e, \cdot) < \kappa_i \\ 1 & \text{if } k'(\kappa_j, e, \cdot) = \kappa_i \\ \frac{\kappa_{i+1} - k'(\kappa_j,e,\cdot)}{\kappa_{i+1} - \kappa_i} & \text{if } \kappa_i < k'(\kappa_j, e, \cdot) < \kappa_{i+1} \\ 0 & \text{if } k'(\kappa_j, e, \cdot) \ge \kappa_{i+1} \end{cases}$$