Introduction to Numerical Methods

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"D", "S", & "GE"

- Dynamic
- Stochastic
- General Equilibrium

What is missing in the abbreviation?

- DSGE models include some form of *forward looking* behavior
  - Typically rational expectations
- Does forward looking makes these models difficult to solve?
  - Yes. How the economy behaves today depends on how agents think the economy will behave in the future for all possible outcomes
  - Agent-based models in which agents predict according to some of pre-specified rules are easier to solve

Recursive problems

Recursive problem

- same *state variables* $\Longrightarrow$ same choices
- Numerous problems are *not* recursive
- Some non-recursive problems can be rewritten as a recursive model by adding state variables
  - e.g., recursive contracts of Marcet & Marrimon

State variables are ...

- the variables that determine the outcomes in period t
  - predetermined variables like the capital stock
  - realizations of exogenous variables, like the productivity shock
  - *not* other endogenous variables to be determined within the period like prices
- Not always trivial to know what the state variables are

Example economy

$$\max_{\{c_{t+j},k_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\nu}-1}{1-\nu}$$

s.t.

$$c_{t+j} + k_{t+j} = z_{t+j}k_{t+j-1}^{\alpha} + (1-\delta)k_{t+j-1}$$
$$z_{t+j} = (1-\rho) + \rho z_{t+j-1} + \varepsilon_t$$
$$k_{t-1} \text{ given}$$

$$\mathbb{E}_{t+j}[\varepsilon_{t+j+1}] = 0 \ \& \ \mathbb{E}_{t+j}[\varepsilon_{t+j+1}^2] = \sigma^2$$

Alternative notation

$$c_t + k_t = z_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}$$

- k_t is the *end*-of-period t capital stock, chosen in t and productive in period $t+1$
- It is also possible that k_t stands for the *beginning*-of-period t capital stock. Then the budget constraint is written as

$$c_t + k_{t+1} = z_t k_t^{\alpha} + (1-\delta)k_t$$

- This is just a change in notation, not in the model

Bellman equation

- If the problem is recursive it can be rewritten using the Bellman equation

$$v(k_{-1}, z) = \max_{c,k} \frac{c^{1-\nu} - 1}{1 - \nu} + \mathbb{E}\left[\beta v(k, z_{+1})\right]$$

$$\text{s.t.}$$

$$c + k = z k_{-1}^{\alpha} + (1 - \delta) k_{-1}$$

First-order conditions

$$
\begin{aligned}
c + k &= z k_{-1}^{\alpha} + (1-\delta) k_{-1} \\
c^{-\nu} &= \beta \mathbb{E}\left[c_{+1}^{-\nu} \left(\alpha z_{+1} k^{\alpha-1} + 1 - \delta \right) \right]
\end{aligned}
$$

Solution have the following form:

$$
\begin{aligned}
c &= c(k_{-1}, z) \\
k &= k(k_{-1}, z)
\end{aligned}
$$

- Why is it difficult to find these solutions?

What should the solutions satisfy?

$$c(k_{-1},z)+k(k_{-1},z)=zk_{-1}^{\alpha}+(1-\delta)k_{-1}$$

$$c(k_{-1},z)^{-\nu}=\beta\mathbb{E}\left[c(k(k_{-1},z),z_{+1})^{-\nu}\left(\alpha z_{+1}k(k_{-1},z)^{\alpha-1}+1-\delta\right)\right]$$

with

$$z_{+1}=(1-\rho)+\rho z+\varepsilon_{+1}$$

Example with analytical solution

- If $\delta = \nu = 1$ then we know the analytical solution. It is

$$k_t = \alpha\beta \exp(z_t) k_{t-1}^a$$
$$c_t = (1 - \alpha\beta) \exp(z_t) k_{t-1}^a$$

or

$$\begin{aligned} \ln k_t &= \ln(\alpha\beta) + \alpha \ln k_{t-1} + z_t \\ \ln c_t &= \ln(1 - \alpha\beta) + \alpha \ln k_{t-1} + z_t \end{aligned}$$

- Solution does not depend on law of motion for z_t. Why not?

State space

- We are going to approximate the policy functions $c(k_{-1}, z)$ and $k(k_{-1}, z)$
- This requires specifying the domain
  - That domain better be bounded $\Longrightarrow$
  - numerical techniques are applied to models that either have no growth or have no growth after transformation of variables

Different ways to numerically solve these models

1. Methods that focus on the first-order conditions
   1. projection methods
   2. perturbation methods
2. Methods that are based on the Bellman equation
   - similar to projection methods
   - often slower, but can handle more complex models (e.g. discontinuities)

References

- Den Haan, W.J., Dynamic optimization problems
  - basic stuff on getting first-order conditions, transversality conditions, state variables, and dynamic programming
- Den Haan, W.J., Equilibrium models
  - continuation but now using equilibrium infinite-horizon (mainly monetary models) and OLG models
- Ljungqvist, L. and T.J. Sargent, 2004, Recursive Macroeconomic Theory
  - describe many DSGE models and their properties
- Stockey, N.L, R.E. Lucas Jr, with E.C. Prescott, 1989, Recursive methods in economic dynamics