# Simulating models with heterogeneous agents

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# Individual agent

- Subject to employment shocks ($\varepsilon_{i,t} \in \{0,1\}$)
- Incomplete markets
  - only way to save is through holding capital
  - borrowing constraint $k_{i,t+1} \geq 0$
- Competitive firms, thus competitive prices
  - $w_t = (1-\alpha)\, z_t \left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha}$
  - $r_t = \alpha z_t \left(\frac{K_t}{\bar{l}L_t}\right)^{\alpha-1}$

# Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty} \beta^t \ln(c_{i,t})$$

s.t.

$$c_{i,t} + k_{i,t+1} = r_t k_{i,t} + (1-\tau_t) w_t \bar{l} \varepsilon_{i,t} + \mu w_t (1-\varepsilon_{i,t}) + (1-\delta) k_{i,t}$$

$$k_{i,t+1} \geq 0$$

# Laws of motion

- $z_t$ can take on two values
- $\varepsilon_{i,t}$ can take on two values
- probability of being (un)employed depends on $z_t$
- transition probabilities are such that unemployment rate only depends on current $z_t$
- Thus $u_t = u^b$ if $z_t = z^b$ and $u_t = u^g$ if $z_t = z^g$ with $u^b > u^g$.

# Complexity of individual problem

- for given process of $r_t$ and $w_t$ this is a relatively simple problem
- state variables?
- constraint?

# What aggregate variables do agents care about?

- $r_t$ and $w_t$
- They only depend on aggregate capital stock and $z_t$
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts $K_t$
- Thus, complete cross-sectional distribution of employment status and capital levels matters

# Equilibrium

- Continuum of agents
- Individual policy functions that solve the agent's maximization problem
- A wage and a rental rate given by equations above.
- A transition law for the cross-sectional distribution of capital, that is consistent with the investment policy function.
  - $f_t$ = beginning-of-period cross-sectional distribution of capital and the employment status after the employment status has been realized.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

# Two different ways to go

- Simulate a panel with a large number of agents
  - This uses Monte Carlo integration to calculate cross-sectional moments
- Use tools from numerical literature
  - grid method that does not require the inverse of the policy function
  - grid method that requires the inverse of the policy function
  - non-grid method

# What is given?

- A policy function $k'(k_{i,t}, \varepsilon_{i,t}, s_t)$
  - $s_t$: the aggregate state variables
- initial distribution for $t = 1$
  - characterizes the density of capital holdings of the employed and unemployed.

# Grid method I

- Fine grid with nodes: $\kappa_i$, $i = 0, 1, \cdots, \chi$
- Only mass *AT* grid points
  - $p^{\varepsilon}_{i,t}$ : mass of agents with $k^{\varepsilon}_t = \kappa_i$, $i = 0, 1, \cdots, \chi$
  - $\varepsilon$ : employment status
  - no mass in between grid points
- If $k'_i \geq 0$ is binding $\Longrightarrow p^{\varepsilon}_{0,t} > 0$ (and CDF has some jumps at other points)

# Grid method I

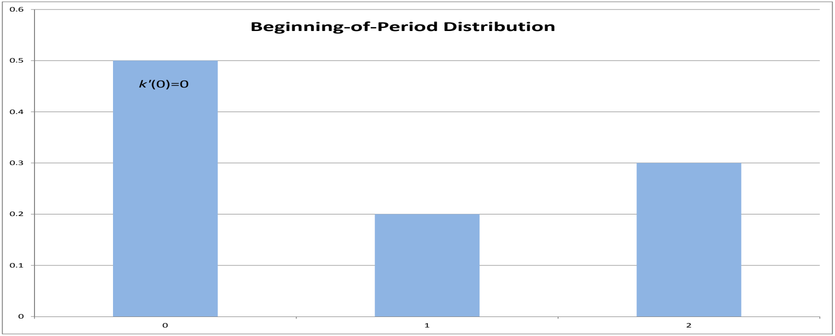
- Fix employment status
  - remain within the period $t$ for now
- Nodes correspond with *beginning-of-period* $t$ distribution

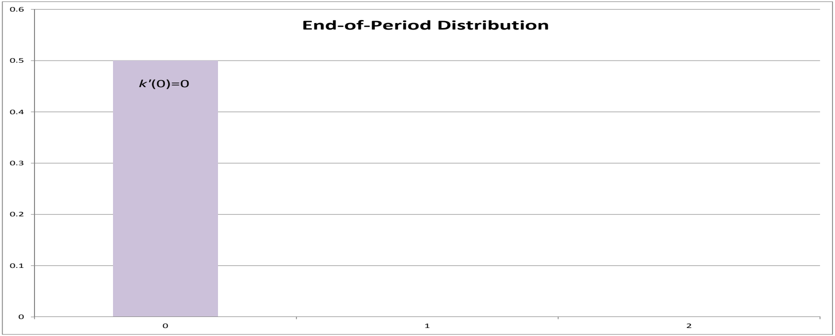
# Grid method I

- focus on node $j$ with mass $p_t^{\varepsilon,j}$ and capital value $\kappa_j$
- find $i$ such that $k'(\kappa_j, \varepsilon, \cdot)$ satisfies

$$\kappa_{i-1} < k'(\kappa_j, \varepsilon, \cdot) \leq \kappa_i$$

- if $k'(\kappa_j, \varepsilon, \cdot) > \kappa_\chi$, $i = \chi$


Beginning-of-Period Distribution
k′(0)=0
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2


End-of-Period Distribution
k′(0)=0
0.6
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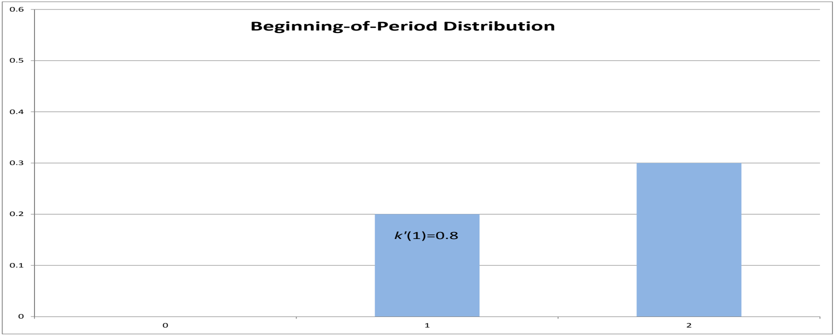
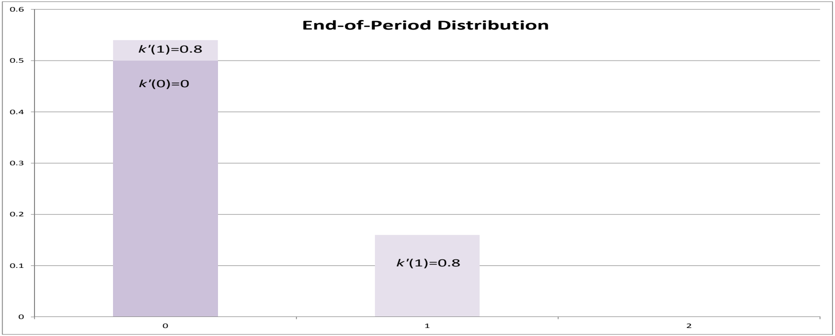

Beginning-of-Period Distribution
0.6
0.5
0.4
0.3
0.2
0.1
0
k′(1)=0.8
0
1
2


End-of-Period Distribution
0.6
0.5
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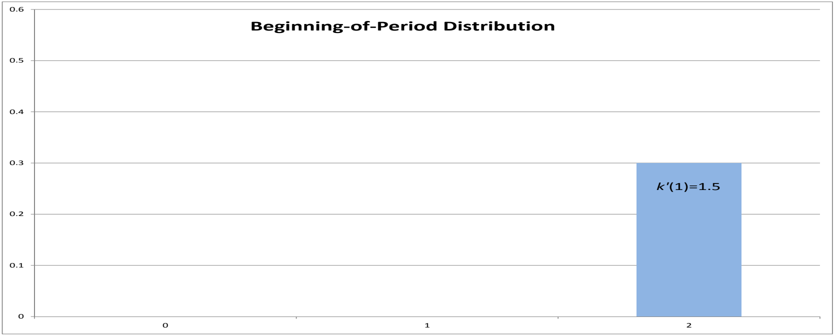
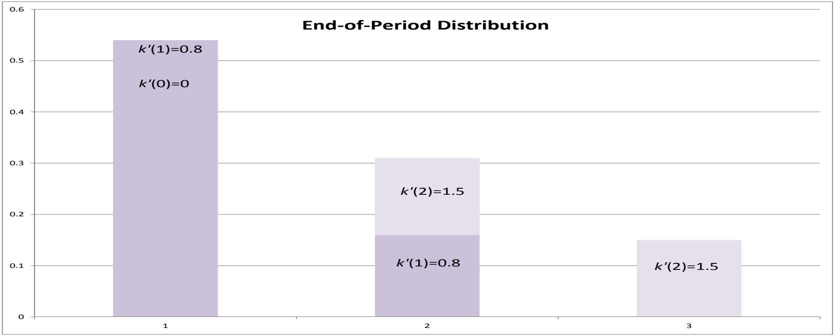
k′(1)=0.8
k′(0)=0
k′(1)=0.8
0
1
2


Beginning-of-Period Distribution
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
k′(1)=1.5


End-of-Period Distribution
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
k′(1)=0.8
k′(0)=0
k′(2)=1.5
k′(1)=0.8
k′(2)=1.5

# Grid method I

- Set end-of-period fractions:

$$f_t^{\varepsilon,i} = 0 \quad \forall i$$

- Go through all nodes $i$ and allocate beginning-of-period $p_t^{\varepsilon,j}$ (for relevant $j$) to end-of-period $f_t^{\varepsilon,i}$:
- if $k'(\kappa_j, \varepsilon, \cdot) < \kappa_\chi$ then

$$\begin{aligned} \omega_t^{i,j} &= \tfrac{k'(\kappa_j,\varepsilon,\cdot)-\kappa_{i-1}}{\kappa_i-\kappa_{i-1}} \\ f_t^{\varepsilon,i-1} &= f_t^{\varepsilon,i-1} + p_t^{\varepsilon,j}\left(1-\omega_t^{i,j}\right) \\ f_t^{\varepsilon,i} &= f_t^{\varepsilon,i} + p_t^{\varepsilon,j}\omega_t^{i,j} \end{aligned}$$

- if $k'(\kappa_j, \varepsilon, \cdot) \geq \kappa_\chi$ then

$$f_t^{\varepsilon,\chi} = f_t^{\varepsilon,\chi} + p_t^{\varepsilon,j}$$

# Grid method I

- Use transition laws to go from end-of-period $t$ to beginning-of-period $t+1$ distribution

# Next period's distribution?

- $g_{\varepsilon_t \varepsilon_{t+1} z_t z_{t+1}}$ : mass of agents with employment status $\varepsilon_t$ that will have employment status $\varepsilon_{t+1}$, conditional on $z_t$ and $z_{t+1}$
- For each combination of values of $z_t$ and $z_{t+1}$ we have

$$g_{00 z_t z_{t+1}} + g_{01 z_t z_{t+1}} + g_{10 z_t z_{t+1}} + g_{11 z_t z_{t+1}} = 1$$

- Fraction of agents with employment status $\varepsilon_t = 0$ that will have employment status $\varepsilon_{t+1} = 0$.

$$\frac{g_{00 z_t z_{t+1}}}{g_{00 z_t z_{t+1}} + g_{01 z_t z_{t+1}}}$$

- The other 3 population moments can be defined the same way.

# Next period's distribution?

Next, we use the population transition probabilities for each node (Since probabilities are exogenous and do not depend on wealth). For $\varepsilon_{t+1} = 0$ we get

$$p_{t+1}^{0,i} = \frac{g_{00z_tz_{t+1}}}{g_{00z_tz_{t+1}} + g_{01z_tz_{t+1}}} f_t^{0,i} + \frac{g_{10z_tz_{t+1}}}{g_{10z_tz_{t+1}} + g_{11z_tz_{t+1}}} f_t^{1,i}$$

For $\varepsilon_{t+1} = 1$ we get

$$p_{t+1}^{1,i} = \frac{g_{01z_tz_{t+1}}}{g_{00z_tz_{t+1}} + g_{01z_tz_{t+1}}} f_t^{0,i} + \frac{g_{11z_tz_{t+1}}}{g_{10z_tz_{t+1}} + g_{11z_tz_{t+1}}} f_t^{1,i}$$

# Grid method II

- Grid method I simply goes through beginning-of-period distribution and allocates each bin to relevant end-of-period bins.
- Grid method II starts at the end-of-period distribution and determines which beginning-of-period bins would end up there. **This requires calculating the inverse of the policy funciton.**

# Grid method II

- Distribution uniformly distributed between grid points
    - CDFs: two piece-wise linear splines, $P_t^{\varepsilon=0}(k)$ and $P_t^{\varepsilon=1}(k)$

# Grid method II

- Calculate the end-of-period distribution as follows
  - nodes correspond to the *end-of-period* distribution
  - go through the nodes, $\kappa_i$, one by one
  - calculate the beginning-of-period capital stock at which the agent would have chosen the value at the grid point, $x_t^{\varepsilon,i} = k^{\prime,inv}(\kappa_i, \varepsilon_t, s_t)$
  - CDF at grid point is equal to $P_t^{\varepsilon}(x_t^{\varepsilon,i})$ (Note the two time subscripts)
  - $$F_t^{\varepsilon,i} = \int_0^{x_t^{\varepsilon,i}} dP_t^{\varepsilon}(k) = \sum_{i=0}^{\overline{i_\varepsilon}} p_t^{\varepsilon,i} + \frac{x_t^{\varepsilon,i} - \kappa_{\overline{i_\varepsilon}}}{\kappa_{1+\overline{i_\varepsilon}} - \kappa_{\overline{i_\varepsilon}}} p_t^{\varepsilon,\overline{i_\varepsilon}+1},$$

    where $\overline{i_\varepsilon} = \bar{i}(x_t^{\varepsilon,i})$ is the largest value of $i$ such that $\kappa_i \leq x_t^{\varepsilon,i}$
- Calculate next period's beginning-of-period distribution using the transtion laws

# Next period's distribution?

Next, we use the population transition probabilities for each node (Since probabilities are exogenous and do not depend on wealth). For $\varepsilon_{t+1} = 0$ we get

$$P^{0,i}_{t+1} = \frac{g_{00z_tz_{t+1}}}{g_{00z_tz_{t+1}} + g_{01z_tz_{t+1}}} F^{0,i}_t + \frac{g_{10z_tz_{t+1}}}{g_{10z_tz_{t+1}} + g_{11z_tz_{t+1}}} F^{1,i}_t$$

For $\varepsilon_{t+1} = 1$ we get

$$P^{1,i}_{t+1} = \frac{g_{01z_tz_{t+1}}}{g_{00z_tz_{t+1}} + g_{01z_tz_{t+1}}} F^{0,i}_t + \frac{g_{11z_tz_{t+1}}}{g_{10z_tz_{t+1}} + g_{11z_tz_{t+1}}} F^{1,i}_t$$

and

$$\begin{aligned} p^{\varepsilon,0}_{t+1} &= P^{\varepsilon,0}_{t+1} \\ p^{\varepsilon,i}_{t+1} &= P^{\varepsilon,i}_{t+1} - P^{\varepsilon,i-1}_{t+1} \end{aligned}$$

# Parameterized cross-sectional distribution

- Grid methods are likely to require a lot of nodes (1000 is typical)
- For some procedures that is costly
- Not clear you need such precise information

# Parameterized cross-sectional distribution

- Grid methods are likely to require a lot of nodes (1000 is typical)
- For some procedures that is costly
- Not clear you need such precise information
- Algan, Allais, and Den Haan (2006) propose to use polynomials
  - $P(k;\rho_t)$ is a polynomial with in period $t = 1$ coefficients equal to $\rho_1$
  - Using Simpson quadrature to calculate end-of-period moments
  - Use transition laws to calculate next period's beginning-of-period moments
- You need a way to find $\rho_2$ given values for moments in period 2

# Fitting a distribution given moments:

- Find $N$ elements of $\rho$ such that

$$\int_0^\infty [k - m(1)] \; P(k;\rho)dk = 0$$

$$\int_0^\infty \left[(k - m(1))^2 - m(2)\right] \; P(k;\rho)dk = 0$$

$$\cdots$$

$$\int_0^\infty \left[(k - m(1))^N - m(N)\right] \; P(k;\rho)dk = 0$$

$$\int_0^\infty P(k;\rho)dk = 1$$

where $m(n)$ is the $n^{\text{th}}$ moment

# Fitting a distribution given moments

- This can be a nasty problem
- But problem made easy by using
- $$P(k;\rho) = \rho_0 \exp\left( \begin{array}{c} \rho_1 \left[k - m(1)\right] + \\ \rho_2 \left[(k - m(1))^2 - m(2)\right] + \cdots + \\ \rho_N \left[(k - m(1))^N - m(N)\right] \end{array} \right)$$

# Fitting a distribution given moments

- For this functional form the coefficients $\rho$ can be found by

$$\min_{\rho_1,\rho_2,\cdots,\rho_N} \int_0^\infty P(k,\rho)dk$$

- The first-order conditions correspond exactly to the condition that the first $N$ moments of $P(k,\rho)$ should correspond to the set of specified moments.
- $\rho_0$ is determined by the condition that the density integrates to one.

# Fitting a distribution given moments

- Knowing the Hessian is useful for optimization routines.
- The Hessian (times $\rho_0$) is given by

$$\int_0^\infty X\left(m(1),\cdots,m(N)\right)X\left(m(1),\cdots,m(N)\right)' P(k,\rho)dk, \quad (1)$$

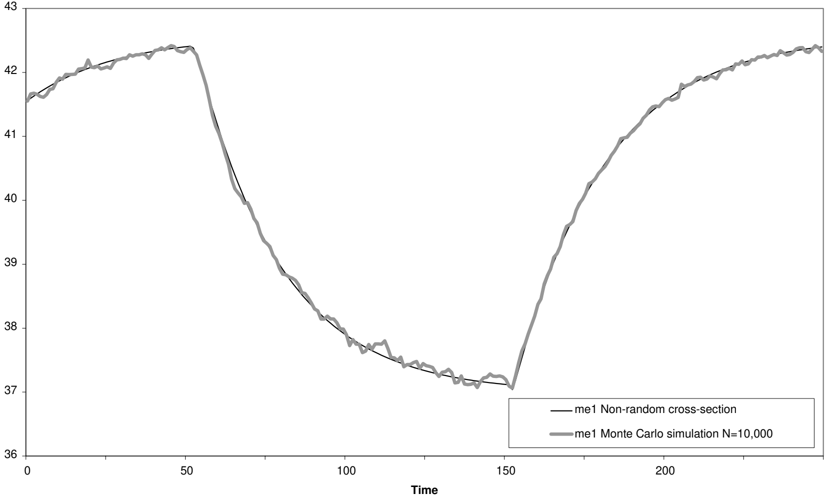
where $X$ is an $(N \times 1)$ vector and the $i^{\text{th}}$ element is given by

$$\begin{array}{cc} (k-m(1)) & \text{for } i=1 \\ (k-m(1))^i - m(i) & \text{for } i>1 \end{array} \quad (2)$$

# Avoiding sampling noise

- What if you really do like to simulate a panel with a finite number of agents?
  - Impose truth as much as possible: if you have 10,000 agents have exactly 400 (1,000) agents unemployed in a boom (recession)
  - Even then sampling noise is non-trivial
  - This is done in the graphs below, but still the results are not accurate

# Simulation and sampling noise



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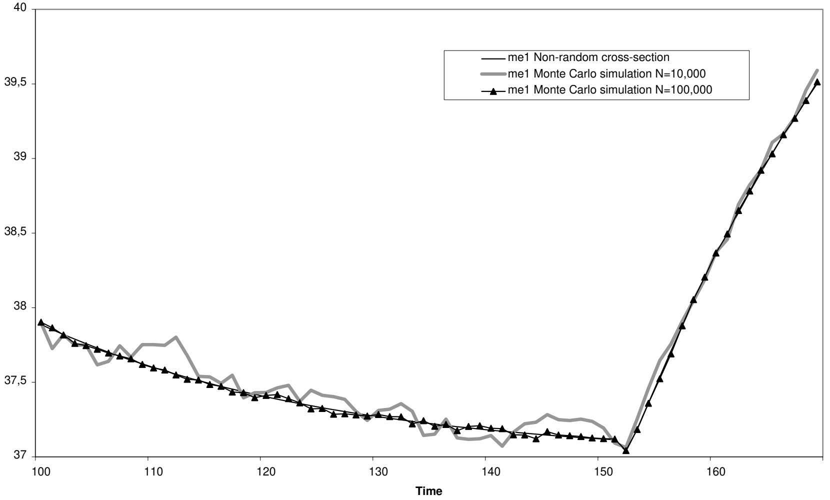

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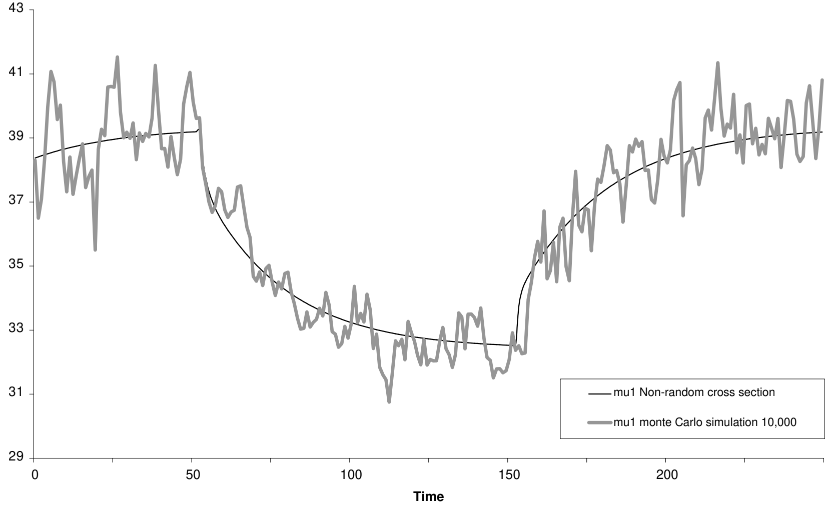

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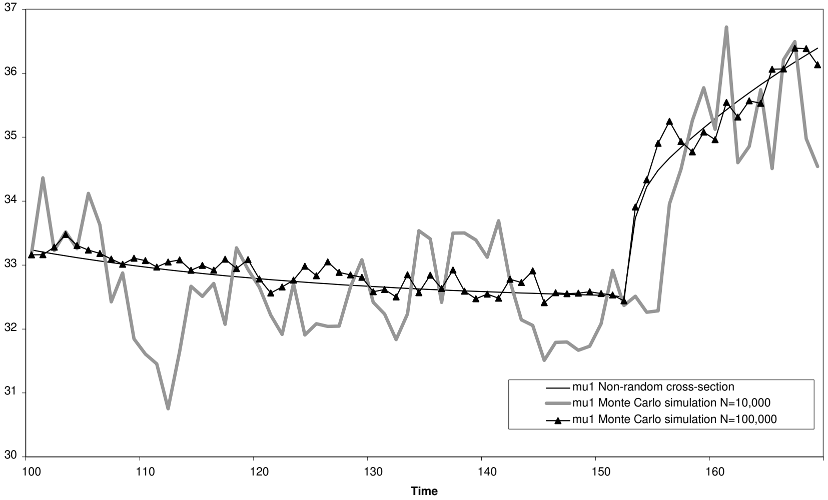

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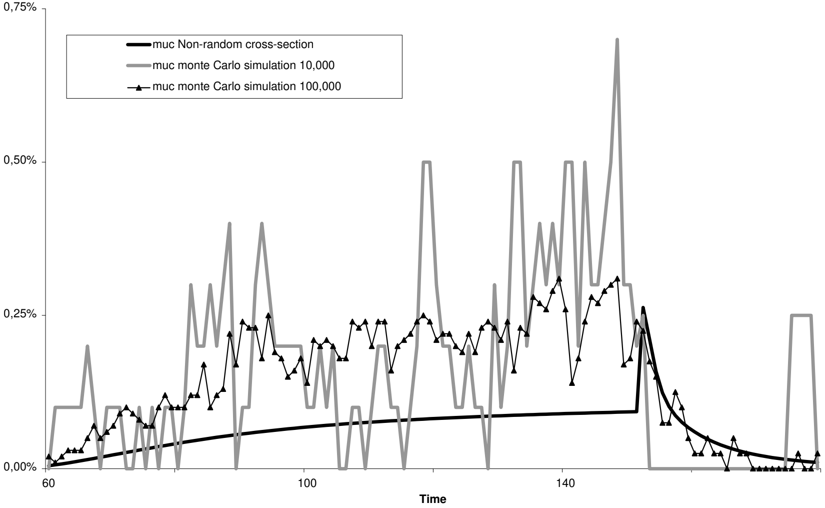

# References


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