Perturbation and LQ

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Neoclassical growth model - no uncertainty

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

s.t.

$$c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t$$

k_1 is given

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right]$$

Neoclassical growth model - no uncertainty

When we substitute out consumption using the budget constraint we get

$$
\begin{gathered}
\left(k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} \\
= \\
\beta\left(k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}\right)^{-\gamma}\left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right],
\end{gathered}
$$

General specification I

$$f(x, x', y, y') = 0.$$

- $x : n_x \times 1$ vector of endogenous & exogenous state variables
- $y : n_y \times 1$ vector of endogenous choice variable

General specification II

Model:

$$f(x'', x', x) = 0$$

for a *known* function $f(\cdot)$.

Solution is of the form:

$$x' = h(x)$$

Thus,

$$F(x) \equiv f(\ h(h(x))\ ,\ h(x)\ , x) = 0 \qquad \forall x$$

Neoclassical growth model again

$$f(k',k,c',c) =$$

$$\left[\begin{array}{c} -c^{-\gamma} + \beta \left(c'\right)^{-\gamma} \left[\alpha \left(k'\right)^{\alpha-1} + 1 - \delta\right] \\ -c - k' + k^{\alpha} + (1-\delta)k \end{array} \right]$$

Solution is of the form:

$$\begin{aligned} k' &= h(k) \\ c &= g(k) \end{aligned}$$

Thus,

$$F(k) \equiv$$

$$\left[\begin{array}{c} -g(k)^{-\gamma} + \beta g(h(k))^{-\gamma} \left[\alpha h(k)^{\alpha-1} + 1 - \delta\right] \\ -g(k) - h(k) + k^{\alpha} + (1-\delta)k \end{array} \right]$$

Neoclassical growth model again

$$f(k'',k',k) = \begin{array}{c} (-k^\alpha - (1-\delta)k - k')^{-\gamma} \\ + \\ \beta\left((k')^\alpha + (1-\delta)k' - k''\right)^{-\gamma}\left[\alpha(k')^{\alpha-1} + 1 - \delta\right], \end{array}$$

for known values of α, δ, and γ

Solution is of the form: $k' = h(k)$
Thus

$$\begin{array}{c} F(k) \equiv \\ (-k^\alpha - (1-\delta)k - h(k))^{-\gamma} + \\ \beta\left(h(k)^\alpha + (1-\delta)h(k) - h(h(k))\right)^{-\gamma}\left[\alpha h(k)^{\alpha-1} + 1 - \delta\right], \end{array}$$

Key condition

$$F(k) = 0 \qquad \forall x$$

Linear, Log-linear, t(x) linear, etc

- All first-order solutions are linear in something
- Specification in last slide
  - $\Longrightarrow$ solution that is linear in the *level* of k

Linear, Log-linear, t(x) linear, etc

- How to get a solution that is linear in $\tilde{k} = \ln(k)$?
- write the $f(\cdot)$ function as

$$f(\tilde{k}'', \tilde{k}', \tilde{k}) = \begin{array}{c} \left(-\exp\left(\alpha\tilde{k}\right) - (1-\delta)\exp\left(\tilde{k}\right) - \exp\left(\tilde{k}'\right)\right)^{-\gamma} \\ + \\ \beta\left(\exp\left(\alpha\tilde{k}'\right) + (1-\delta)\exp\left(\tilde{k}'\right) - \exp\left(\tilde{k}''\right)\right)^{-\gamma} \\ \times \\ \left[\alpha\exp\left((\alpha-1)\tilde{k}'\right) + 1 - \delta\right] \end{array}$$

Linear, Log-linear, t(x) linear, etc

- How wo get a solution that is linear in $\hat{k} = t(k)$?
- Write the $f(\cdot)$ function as

$$f(\hat{k}'', \hat{k}', \hat{k}) \\ = \\ (-\left(t_{inv}(\hat{k})\right)^{\alpha} - (1-\delta)\left(t_{inv}(\hat{k})\right) - \left(t_{inv}(\hat{k}')\right))^{-\gamma} + \\ \beta\left(\left(t_{inv}(\hat{k}')\right)^{\alpha} + (1-\delta)\left(t_{inv}(\hat{k}')\right) - \left(t_{inv}(\hat{k}'')\right)\right)^{-\gamma} \times \\ \left[\alpha\left(t_{inv}(\hat{k}')\right)^{\alpha-1} + 1 - \delta\right]$$

Numerical solution

Let

$$\bar{x} \text{ solve } f(\bar{x}, \bar{x}, \bar{x}) = 0$$

That is

$$\bar{x} = h(\bar{x})$$

Taylor expansion

$$\begin{aligned} h(x) &\approx h(\bar{x}) + (x - \bar{x})h'(\bar{x}) + \frac{(x - \bar{x})^2}{2} h''(\bar{x}) + \cdots \\ &= \bar{x} + \bar{h}_1 (x - \bar{x}) + \bar{h}_2 \frac{(x - \bar{x})^2}{2} + \cdots \end{aligned}$$

- Goal is to find $\bar{x}$, $\bar{h}_1$, $\bar{h}_2$, etc.

Solving for first-order term

$$F(x) = 0 \quad \forall x$$

Implies

$$F'(x) = 0 \quad \forall x$$

Definitions

Let

$$
\begin{aligned}
\left.\frac{\partial f(x'',x',x)}{\partial x''}\right|_{x''=x'=x=\bar{x}} &= \bar{f}_1, \\
\left.\frac{\partial f(x'',x',x)}{\partial x'}\right|_{x''=x'=x=\bar{x}} &= \bar{f}_2, \\
\left.\frac{\partial f(x'',x',x)}{\partial x}\right|_{x''=x'=x=\bar{x}} &= \bar{f}_3.
\end{aligned}
$$

Note that

$$
\left.\frac{\partial h(x)}{\partial x}\right|_{x=\bar{x}} = \left(\bar{h}_1 + \bar{h}_2(x-\bar{x}) + \cdots\right)\Big|_{x=\bar{x}} = \bar{h}_1
$$

Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\overline{x}) = \overline{f}_1\overline{h}_1^2 + \overline{f}_2\overline{h}_1 + \overline{f}_3 = 0$$

- One equation to solve for $\overline{h}_1$

Key equation

$$F'(x) = 0 \quad \forall x$$

or

$$F'(x) = \frac{\partial f}{\partial x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x} = 0$$

can be written as

$$F'(\bar{x}) = \bar{f}_1 \bar{h}_1^2 + \bar{f}_2 \bar{h}_1 + \bar{f}_3 = 0$$

- One equation to solve for $\bar{h}_1$
- Hopefully, the Blanchard-Kahn conditions are satisfied and there is only one sensible solution

Solving for second-order term

$$F'(x) = 0 \quad \forall x$$

Implies

$$F''(x) = 0 \quad \forall x$$

Definitions

Let

$$\left.\frac{\partial^2 f(x'', x', x)}{\partial x'' \partial x}\right|_{x''=x'=x=\bar{x}} = \bar{f}_{13}. \tag{1}$$

and note that

$$\left.\frac{\partial^2 h(x)}{\partial x^2}\right|_{x=\bar{x}} = \left.\left(\bar{h}_2 + \bar{h}_3(x-\bar{x}) + \cdots\right)\right|_{x=\bar{x}} = \bar{h}_2. \tag{2}$$

Key equation

$$F''(x) = 0 \quad \forall x$$

or

$$F''(x) =$$

$$+ \left(\frac{\partial^2 f}{\partial x''^2} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'' \partial x} \right) \left(\frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} \right)$$

$$+ \frac{\partial f}{\partial x''} \left(\frac{\partial h(x')}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2} + \frac{\partial^2 h(x')}{\partial x'^2} \frac{\partial h(x)}{\partial x} \frac{\partial h(x)}{\partial x} \right)$$

$$+ \left(\frac{\partial^2 f}{\partial x' x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'^2} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x' \partial x} \right) \frac{\partial h(x)}{\partial x}$$

$$+ \frac{\partial f}{\partial x'} \frac{\partial^2 h(x)}{\partial x^2}$$

$$+ \left(\frac{\partial^2 f}{\partial x x''} \frac{\partial h(x')}{\partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x \partial x'} \frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x^2} \right)$$

Key equation

Which can be written as

$$F''(\bar{x}) = \left(\bar{f}_{11}\bar{h}_1^2 + \bar{f}_{12}\bar{h}_1 + \bar{f}_{13}\right)\bar{h}_1^2 + \bar{f}_1(\bar{h}_1\boldsymbol{\bar{h}_2} + \boldsymbol{\bar{h}_2}\bar{h}_1^2)$$
$$+$$
$$\left(\bar{f}_{21}\bar{h}_1^2 + \bar{f}_{22}\bar{h}_1 + \bar{f}_{23}\right)\bar{h}_1 + \bar{f}_2\boldsymbol{\bar{h}_2} + \left(\bar{f}_{31}\bar{h}_1^2 + \bar{f}_{32}\bar{h}_1 + \bar{f}_{33}\right) = 0$$

- One *linear* equation to solve for $\bar{h}_2$

Discussion

- Global or local?
- Borrowing constraints?
- Quadratic/cubic adjustment costs?

Neoclassical growth model with uncertainty

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathsf{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

s.t.

$$c_t + k_{t+1} = \exp(\theta_t) k_t^{\alpha} + (1-\delta) k_t \tag{3}$$

$$\theta_t = \rho \theta_{t-1} + \sigma e_t, \tag{4}$$

General specification

$$\mathsf{E}f(x, x', y, y') = 0.$$

- $x : n_x \times 1$ vector of endogenous & exogenous state variables
- $y : n_y \times 1$ vector of endogenous choice variable
- Stochastic growth model: $y = c$ and $x = [k, \theta]$.

Essential insight #1

- Make uncertainty (captured by *one* parameter) explicit in system of equation

$$\mathsf{E}f(x, x', y, y', \sigma) = 0.$$

Solutions are of the form:

$$y = g(x, \sigma)$$

and

$$x' = h(x, \sigma) + \sigma \eta \varepsilon'$$

Neoclassical Growth Model

- For standard growth model we get

$$\mathrm{E} f([k,\theta],[k',\rho\theta+\sigma\varepsilon'],y,y')=0$$

Solutions are of the form:

$$c = c(k,\theta,\sigma) \tag{5}$$

and

$$\begin{bmatrix} k' \\ \theta' \end{bmatrix} = \begin{bmatrix} k'(k,\theta,\sigma) \\ \rho\theta \end{bmatrix} + \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} e'. \tag{6}$$

Essential insight #2

Perturb around y, x, and σ.

$$g(x,\sigma) = g(\overline{x},0) + g_x(\overline{x},0)(x-\overline{x}) + g_\sigma(\overline{x},0)\sigma + \cdots$$

and

$$h(x,\sigma) = h(\overline{x},0) + h_x(\overline{x},0)(x-\overline{x}) + h_\sigma(\overline{x},0)\sigma + \cdots$$

Goal

Let

$$\bar{g}_x = g_x(\bar{x}, 0),\ \bar{g}_\sigma = g_\sigma(\bar{x}, 0) \text{ and}$$
$$\bar{h}_x = h_x(\bar{x}, 0),\ \bar{h}_\sigma = h_\sigma(\bar{x}, 0).$$

Goal: to find

- $(n_y \times n_x)$ matrix $\bar{g}_x$, $(n_y \times 1)$ vector $\bar{g}_\sigma$, $(n_x \times n_x)$ matrix $\bar{h}_x$, $(n_x \times 1)$ vector $\bar{h}_\sigma$.
- The total of unknowns $=$ $(n_x + n_y) \times (n_x + 1) = n \times (n_x + 1)$.

More on uncertainty

Results for first-order perturbation

- $\bar{g}_\sigma = \bar{h}_\sigma = 0$

Results for second-order perturbation

- $\bar{g}_{\sigma x} = \bar{h}_{\sigma x} = 0$, but $\bar{g}_{\sigma\sigma} \neq 0$ and $\bar{h}_{\sigma\sigma} \neq 0$

How to model discrete support?

Theory

- If the function is analytical $\Longrightarrow$ successive approximations converge towards the truth
- Theory says nothing about convergence patterns
- Theory doesn't say whether second-order is better than first
- For complex functions, this is what you have to worry about

Example with simple Taylor expansion

Truth:

$$
\begin{aligned}
f(x) &= -690.59 + 3202.4x - 5739.45x^2 \\
&\quad +4954.2x^3 - 2053.6x^4 + 327.10x^5
\end{aligned}
$$

defined on $[0.7, 2]$

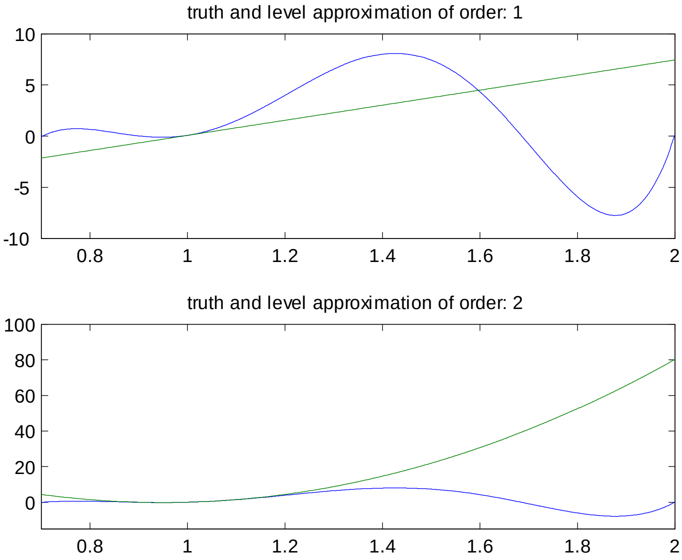


Figure: Level approximations

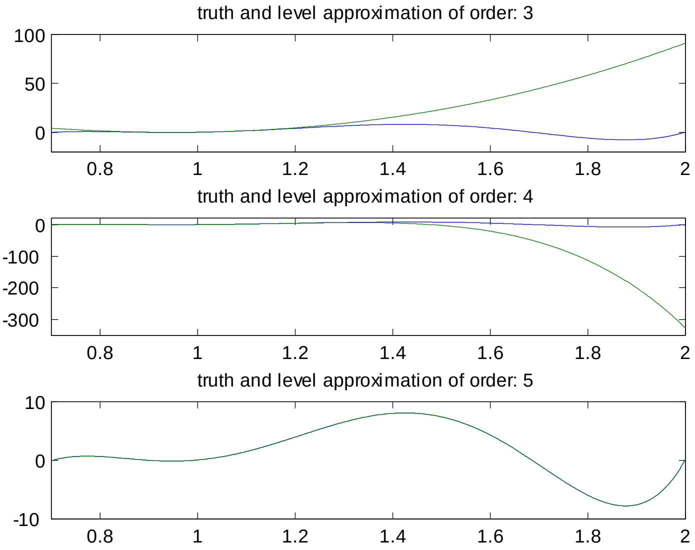


Figure: Level approximations continued

Approximation in log levels

Think of $f(x)$ as a function of $z = \log(x)$. Thus,

$$\begin{aligned} f(x) &= -690.59 + 3202.4\exp(z) - 5739.45\exp(2z) \\ &\quad + 4954.2\exp(3z) - 2053.6\exp(4z) + 327.10\exp(5z). \end{aligned}$$

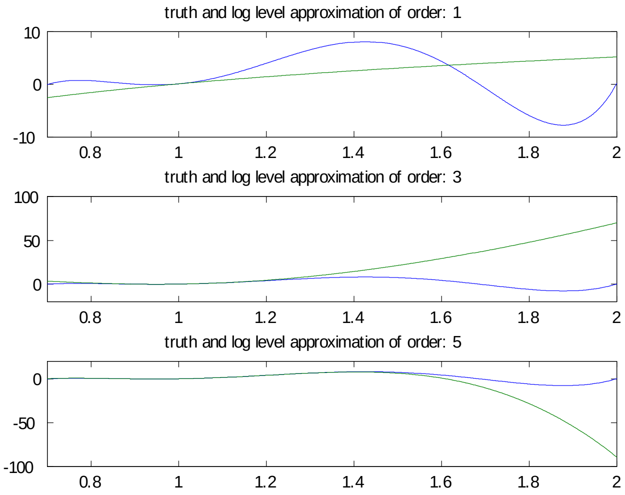


Figure: Log level approximations

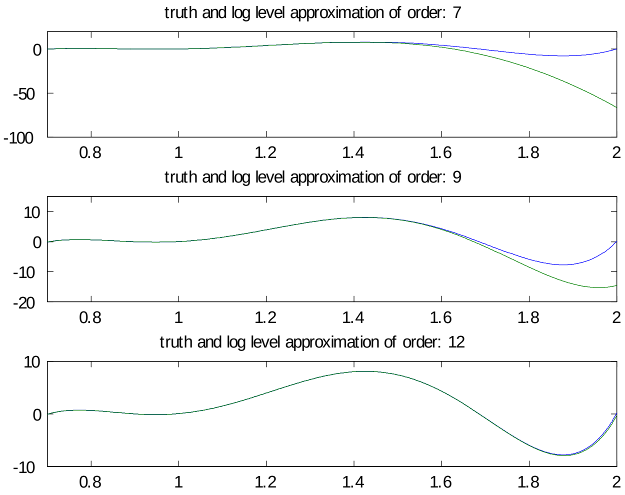
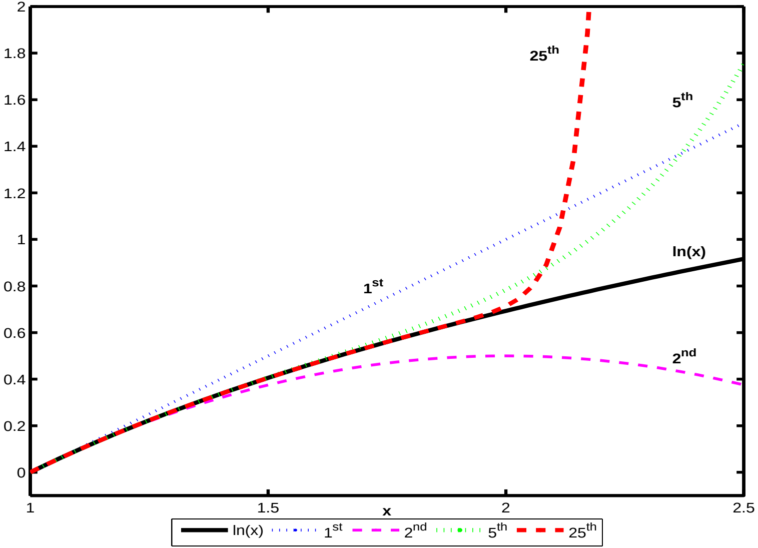


Figure: Log level approximations continued

ln(x) & Taylor series expansions at x = 1



Are LQ & first-order perturbation the same?

True model:

$$\max_{x,y} f(x,y,a)$$
$$\text{s.t.} \;\; g(x,y,a) \leq b$$

First-order conditions

$$\begin{aligned} f_x(x,y,a) + \lambda g_x(x,y,a) &= 0 \\ f_y(x,y,a) + \lambda g_y(x,y,a) &= 0 \\ g(x,y,a) &= b \end{aligned}$$

- First-order perturbation of this system will involve second-order derivatives of $g(\cdot)$
- LQ solution will not

Benigno and Woodford LQ approach

Basic Idea: Add second-order approximation to objective function
Naive LQ approximation:

$$\max_{x,y}\min_{\lambda} \begin{array}{c} +\ \bar{f}_x\tilde{x}+\bar{f}_y\tilde{y}+\bar{f}_a\tilde{a} \\ +\frac{1}{2}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{a}\end{bmatrix}' \begin{bmatrix}\bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xa}\\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{ya}\\ \bar{f}_{ax} & \bar{f}_{ay} & \bar{f}_{aa}\end{bmatrix} \begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{a}\end{bmatrix} \\ +\lambda\left[-\bar{g}_x\tilde{x}-\bar{g}_y\tilde{y}-\bar{g}_a\tilde{a}\right] \end{array} \tag{7}$$

Benigno and Woodford LQ approach

Step I:Take second-order approximation of constraint.

$$0 \approx \begin{array}{c} \bar{g}_x \tilde{x} + \bar{g}_y \tilde{y} + \bar{g}_a \tilde{a} \\ +\frac{1}{2} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix}' \begin{bmatrix} \bar{g}_{xx} & \bar{g}_{xy} & \bar{g}_{xa} \\ \bar{g}_{yx} & \bar{g}_{yy} & \bar{g}_{ya} \\ \bar{g}_{ax} & \bar{g}_{ay} & \bar{g}_{aa} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{a} \end{bmatrix} \end{array} \tag{8}$$

Benigno and Woodford LQ approach

Step 2:Multiply by steady state value of λ and add to "naive" LQ formulation:

$$\max_{x,y}\min_{\lambda}\frac{1}{2}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}'\begin{bmatrix}\bar{f}_{xx}-\bar{\lambda}\bar{g}_{xx} & \bar{f}_{xy}-\bar{\lambda}\bar{g}_{xy} & \bar{f}_{xa}-\bar{\lambda}\bar{g}_{xy}\\ \bar{f}_{yx}-\bar{\lambda}\bar{g}_{yx} & \bar{f}_{yy}-\bar{\lambda}\bar{g}_{yy} & \bar{f}_{ya}-\bar{\lambda}\bar{g}_{ya}\\ \bar{f}_{ax}-\bar{\lambda}\bar{g}_{ax} & \bar{f}_{ay}-\bar{\lambda}\bar{g}_{ay} & \bar{f}_{aa}-\bar{\lambda}\bar{g}_{aa}\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}$$
$$+\lambda\left[b-\bar{g}-\bar{g}_x\widetilde{x}-\bar{g}_y\widetilde{y}-\bar{g}_a\widetilde{a}\right]$$