3
Perturbation techniques

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3.1 Introduction

In this set of notes we show how perturbation techniques can be used to obtain first and higher-order Taylor expansions of the true rational expectations policy function around the steady state. We also discuss the paper of Schmitt-Grohé and Uribe (2004) that makes clear in which way uncertainty affects the policy rules obtained with pertubation solutions.

We also make clear what the difference is between the first-order approximation obtained with the perturbation procedure and the first-order approximation obtained with what Benigno and Woodford (2006) refer to as the naive LQ procedure. This is the linear solution one obtains using a quadratic approximation of the objective function and a *linear* approximation of the constraints. This LQ procedure does *not* generate in general the first-order Taylor expansion of the true rational expectations solution. The reason is that the constraints are only approximated with first-order approximations. The rational expectations solution is itself based on first-order conditions and so the correct first-order Taylor expansion of the true policy rule includes second-order aspects of the objective function as well as the constraints. Moreover, one cannot use a second-order approximation of the constraints because the solution would be no longer linear and the whole convenience of the LQ framework disappears.[1]

This result implies that it is better to get rid of the constraints by substituting out variables. This is not always possible. Benigno and Woodford (2006) show that one can incorporate second-order properties of the constraints into the Lagrangian and still have a standard LQ problem. Using a simple example, we show why this procedure also results in a first-order Taylor expansion of the true solution.

There are no new results in this note. Also, in stead of giving proofs for general formulation we document properties using simple examples. Hopefully this will make the ideas easier to understand. Also, these notes are new and the expressions get a bit tedious so be aware of typos and if you find them please let me know.

3.2 Case without uncertainty

Consider the standard growth model.

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$
$$\text{s.t.} \quad c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t$$
$$k_1 \text{ is given.}$$

The Euler equation is given by

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma}\left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right]. \tag{3.1}$$

When we substitute out consumption using the budget constraint we get

$$\begin{aligned} &\left(k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right)^{-\gamma} = \\ &\beta\left(k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}\right)^{-\gamma}\left[\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right], \end{aligned} \tag{3.2}$$

that is, a second-order difference equation in k_t. We are looking for a recursive solution of the form

$$k_{t+1} = h(k_t). \tag{3.3}$$

More generally, we are looking for a solution to equations like

$$f(x'', x', x) = 0 \tag{3.4}$$

[1] If there are second-order terms in the constraint then the Lagrangian would contain third-order terms, namely the Lagrange multiplier times these second-order terms.

of the form

$$x' = h(x). \tag{3.5}$$

To simplify the notation we let (for now) x be a scalar. Define $F(x)$ as

$$F(x) \equiv f(h(h(x)), h(x), x). \tag{3.6}$$

Since $h(x)$ is a solution to Equation (3.4), we know that

$$F(x) = 0. \tag{3.7}$$

Let $\overline{x}$ be the fixed-point of $h(x)$. Thus,

$$\overline{x} = h(\overline{x}). \tag{3.8}$$

The Taylor expansion of the solution, $h(x)$, around $\overline{x}$ is given by

$$h(x) \approx h(\overline{x}) + (x-\overline{x})h'(\overline{x}) + \frac{(x-\overline{x})^2}{2}h''(\overline{x}) + \cdots \tag{3.9}$$

$$= \overline{x} + \overline{h}_1(x-\overline{x}) + \overline{h}_2\frac{(x-\overline{x})^2}{2} + \cdots \tag{3.10}$$

So the goal is to find $\overline{x}$, $\overline{h}_1$, $\overline{h}_2$, etc..

Clearly, $\overline{x}$ has to satisfy

$$f(\overline{x}, \overline{x}, \overline{x}) = 0. \tag{3.11}$$

Finding $\overline{x}$ can be a non-trivial problem if f is a nasty non-linear function, but a good equation solver combined with some decent initial conditions should do the trick. The key insight of the perturbation procedure is to solve for the coefficients $\overline{h}_i$ not simultaneously but sequentially. So let's start.

3.2.1 *Finding the coefficient for the linear term,* $\overline{h}_1$

For what follows below, it is important to understand that the functional form of f and numerical values of parameter values that appear in f are known. Since

$$F(x) = 0 \; \forall x \tag{3.12}$$

we know that

$$F'(x) = 0 \; \forall x. \tag{3.13}$$

The derivative of F is given by

$$F'(x) = \frac{\partial f}{\partial x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial f}{\partial x}. \tag{3.14}$$

Let

$$\left.\frac{\partial f(x'',x',x)}{\partial x''}\right|_{x''=x'=x=\overline{x}} = \overline{f}_1, \tag{3.15}$$

$$\left.\frac{\partial f(x'',x',x)}{\partial x'}\right|_{x''=x'=x=\overline{x}} = \overline{f}_2, \tag{3.16}$$

$$\left.\frac{\partial f(x'',x',x)}{\partial x}\right|_{x''=x'=x=\overline{x}} = \overline{f}_3. \tag{3.17}$$

Also, note that

$$\left.\frac{\partial h(x)}{\partial x}\right|_{x=\overline{x}} = \left(\overline{h}_1 + \overline{h}_2(x-\overline{x}) + \cdots\right)\Big|_{x=\overline{x}} = \overline{h}_1 \tag{3.18}$$

Using this in Equation (3.14) we get

$$F'(\overline{x}) = \overline{f}_1\overline{h}_1^2 + \overline{f}_2\overline{h}_1 + \overline{f}_3 = 0 \tag{3.19}$$

Note that there are no approximations in obtaining this equation. That is, the first-order term of the Taylor expansion of the true policy function is exactly pinned down by this equation. Solving this quadratic equation for $\overline{h}_1$ corresponds to the standard problem of obtaining a solution from the linearized first-order conditions. See, for example, the notes of Harald Uhlig for a discussion. The concavity of the utility and the production function implies that one solution corresponds to an explosive time path and that the other root corresponds to the unique non-explosive solution of the system.

3.2.2 Finding the coefficient for the second-order term, $\overline{h}_2$

Given the solution for $\overline{h}_1$ it is actually relatively simple to get the second-order term. Let's calculate $F''(x)$ by differentiating the expresssion for $F'(x)$ in Equation (3.14).

$$\begin{aligned} F''(x) = & \\ & + \left(\frac{\partial^2 f}{\partial x''^2}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x''\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x''\partial x}\right)\left(\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x}\right) \\ & + \frac{\partial f}{\partial x''}\left(\frac{\partial h(x')}{\partial x'}\frac{\partial^2 h(x)}{\partial x^2} + \frac{\partial^2 h(x')}{\partial x'^2}\frac{\partial h(x)}{\partial x}\frac{\partial h(x)}{\partial x}\right) \\ & + \left(\frac{\partial^2 f}{\partial x' x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'^2}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x'\partial x}\right)\frac{\partial h(x)}{\partial x} \\ & + \frac{\partial f}{\partial x'}\frac{\partial^2 h(x)}{\partial x^2} \\ & + \left(\frac{\partial^2 f}{\partial x x''}\frac{\partial h(x')}{\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x\partial x'}\frac{\partial h(x)}{\partial x} + \frac{\partial^2 f}{\partial x^2}\right) \end{aligned} \tag{3.20}$$

Now you may think this is a nightmare, but it actually isn't. First, we use subscripts to indicate second-order derivatives evaluated at the steady state. For example,

$$\left.\frac{\partial^2 f(x'', x', x)}{\partial x'' \partial x}\right|_{x''=x'=x=\overline{x}} = \overline{f}_{13}. \tag{3.21}$$

Also,

$$\left.\frac{\partial^2 h(x)}{\partial x^2}\right|_{x=\overline{x}} = \left(\overline{h}_2 + \overline{h}_3(x - \overline{x}) + \cdots\right)\Big|_{x=\overline{x}} = \overline{h}_2. \tag{3.22}$$

Combining this gives

$$\begin{aligned} F''(\overline{x}) &= 0 \\ &= \left(\overline{f}_{11}\overline{h}_1^2 + \overline{f}_{12}\overline{h}_1 + \overline{f}_{13}\right)\overline{h}_1^2 \\ &\quad +\overline{f}_1(\overline{h}_1\overline{h}_2 + \overline{h}_2\overline{h}_1^2) \\ &\quad + \left(\overline{f}_{21}\overline{h}_1^2 + \overline{f}_{22}\overline{h}_1 + \overline{f}_{23}\right)\overline{h}_1 \\ &\quad +\overline{f}_2\overline{h}_2 \\ &\quad + \left(\overline{f}_{31}\overline{h}_1^2 + \overline{f}_{32}\overline{h}_1 + \overline{f}_{33}\right) \end{aligned} \tag{3.23}$$

This equation is linear in the only unknown, $\overline{h}_2$, so this is an easy equation to solve. Obtaining higher-order terms can be done by repeating this procedure. And all higher-order terms can be solved from a linear system.

3.3 Is it just a local procedure?

To better understand the formal ideas behind perturbation techniques you should check Judd (1998). But the basic idea is the implicit-function theorem. That is,

1. if $H(x, y) : R^n \times R^m \longrightarrow R^m$,
2. $H_y(x, y)$ is not singular,
3. $H(\overline{x}, \overline{y}) = 0$, and
4. you can differentiate this function sufficiently often,
5. then there exists a unique function $y = h(x)$ such that $H(x, h(x)) = 0$ and the derivatives of h can be obtained by implicit differentiation.

You may think that perturbation procedures can only provide local approximations and that these techniques are not very good in evaluating the policy functions at values of the state variables that are not close to the steady state. That was my feeling when I started thinking about perturbation techniques. It is important to realize though that for a smooth and sufficiently differentiable function $d(z)$, one can approximate $d(z^*)$ well using a Taylor expansion around $\overline{z}$ even though one only uses information about d at a point that is far away from z^*, namely $\overline{z}$. The reason is that for a regular function the functional form of d at z^* is also present in the derivatives of d at $\overline{z}$. For example, suppose that $d(z)$ is a 10^{th}-order polynomial. The value of d at $\overline{z}$ together with the 10 derivatives at $\overline{z}$ completely pin down the function. The 10^{th}-order Taylor expansion, thus, would give a perfect approximation for any value z no matter how far away from $\overline{z}$. The story, of course, breaks down if there are non-differentiabilities. Also, in practice the question is whether *low*-order perturbation methods are accurate and how they compare with low-order approximations obtained from global numerical solution procedures. For example, suppose that the true policy rule is given by $d(z) = z^{10}$ and $\overline{z} = 0$ then anything below a 10^{th}-order perturbation would result in a flat policy function, whereas the truth is not flat.

The following numerical example, documents this. It also points out, however, that convergence towards the truth as the order of the polynomial increases can display very strange patterns. The function considered is a 5^{th}-order polynomial equal to

$$\begin{aligned} f(x) &= -690.59 + 3202.4x - 5739.45x^2 \\ &\quad +4954.2x^3 - 2053.6x^4 + 327.10x^5 \end{aligned}$$

defined on the interval $[0.7, 2]$. The five panels of Figure 1 plot the the true function and the Taylor approximations around $x = 1$ from the first–order to the fifth-order. This function shows show wild osciallations, but the fifth-order Taylor expansion is identical to the truth. Interestingly, of the other approximations the first-order is the best and the fourth-order is the worst. Note that the scale of the vertical axis is very different in each of the five panels.

This raises the question, what convergence would look like if one would use a different type of polynomial. For example, suppose we think of $f(x)$ as a function of $z = \log(x)$. Thus,

$$\begin{aligned} f(x) &= -690.59 + 3202.4\exp(z) - 5739.45\exp(2z) \\ &\quad +4954.2\exp(3z) - 2053.6\exp(4z) + 327.10\exp(5z). \end{aligned}$$

The six panels of Figure 2 plot the Taylor expansions around $z = 0$ of order 1, 3, 5, 7, 9, and 12. Again convergence displays an odd pattern with the approximation actually first getting worse if one goes beyond first-order and only around the 9^{th}-order approximation does the approximation start to resemble the truth and converges monotonically towards it. But note that even for the 7^{th}-order approximation the deviation from the truth is huge (note the difference in the scale).

3.4 The case with uncertainty

Consider the standard growth model with uncertainty:

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$\text{s.t.} \quad c_t + k_{t+1} = \exp(\theta_t) k_t^{\alpha} + (1-\delta) k_t \tag{3.24}$$

where θ_t is a stochastic productivity shock and the initial capital stock k_1 is given. A typical law of motion for θ_t is given by

$$\theta_t = \rho\theta_{t-1} + \sigma e_t, \tag{3.25}$$

where σ controls the amount of uncertainty.

The Euler equation is given by

$$c_t^{-\gamma} = \beta \mathrm{E}_t \left[c_{t+1}^{-\gamma} \left(\alpha \exp(\theta_t) k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]. \tag{3.26}$$

Equations (3.24), (3.25), and (3.26) give a system of three equations that determine the law of motion for consumption, productivity, and capital. Such a system can be written as

$$Ef(x, x', y, y') = 0. \tag{3.27}$$

Here x is an $(n_x \times 1)$ vector of endogenous and exogenous state variables and y is an $(n_y \times 1)$ vector endogenous choice variable. When applied to the stochastic growth model, we would have that $y = c$ and $x = [k, \theta]$.

Essential ingredients

When solving the model with perturbation techniques, we write the problem such that the amount of uncertainty is controlled by one *scalar* parameter σ. Even if there are multiple stochastic driving processes one can still use one parameter to scale the amount of uncertainty. If $\sigma = 0$ then there is no uncertainty. Again, the goal is to find the policy functions. To apply the perturbation technique under uncertainty we follow the following two key steps.

First key step: solution as a function of σ

The first step is to make explicit that σ enters the system of equations and that the policy functions, thus, depends on the amount of uncertainty. For the standard growth model we have

$$Ef([k,\theta],[k',\rho\theta+\sigma\varepsilon'],y,y')=0. \tag{3.28}$$

We are trying to solve for functions of the form

$$y=g(x,\sigma) \tag{3.29}$$

and

$$x'=h(x,\sigma)+\sigma\eta\varepsilon', \tag{3.30}$$

where ε' is an $(n_\varepsilon\times 1)$ vector and η is an $(n_x\times n_\varepsilon)$ matrix. The policy functions depend, of course, on all the structural parameter values, not just σ. But you will see below why we make explicit that it depends on the amount of uncertainty. For example, for the standard growth model presented above, these equations would be

$$c=c(k,\theta,\sigma) \tag{3.31}$$

and

$$\begin{bmatrix} k' \\ \theta' \end{bmatrix}=\begin{bmatrix} k'(k,\theta,\sigma) \\ \rho\theta \end{bmatrix}+\sigma\begin{bmatrix} 0 \\ 1 \end{bmatrix}e'. \tag{3.32}$$

Second key step: perturb around $\overline{y}$ and $\overline{x}$ and $\sigma=0$

The second key step of the perturbation procedure is to take a Taylor expansion of the true solution to the system around the steady state values of the variables *and* around $\sigma=0$. That is, one starts at the steady state but then allows uncertainty to increase.

A disadvantage of perturbation techniques is that the notation is a bit tedious. Below, I will show you the notation that the literature has used and show how to do perturbation under uncertainty. Don't worry if you get lost in the notation. Below, I will go back to the case of the standard growth model and redo the analysis.

Let $\overline{x}=h(\overline{x},\sigma)$ and $\overline{y}=g(\overline{x},\sigma)$. Thus,

$$f(\overline{x},\overline{x},\overline{y},\overline{y})=0 \tag{3.33}$$

Define

$$\begin{gathered} F(x,\sigma)= \\ \mathrm{E}_t\ f\ [x,x',y,y']=0 \\ \mathrm{E}_t\ f\ [x,h(x,\sigma)+\sigma\eta\varepsilon',g(x,\sigma),g(x',\sigma)]=0 \\ \mathrm{E}_t\ f\ [x,h(x,\sigma)+\sigma\eta\varepsilon',g(x,\sigma),g(h(x,\sigma)+\sigma\eta\varepsilon',\sigma)]=0 \end{gathered} \tag{3.34}$$

This is a system of $n(= n_x+n_y)$ in n unknowns. Since we are also perturbing σ (around 0), the Taylor expansions of the true policy functions are given by

$$g(x,\sigma) = g(\overline{x},0) + g_x(\overline{x},0)(x-\overline{x}) + g_\sigma(\overline{x},0)\sigma + \cdots \tag{3.35}$$

and

$$h(x,\sigma) = h(\overline{x},0) + h_x(\overline{x},0)(x-\overline{x}) + h_\sigma(\overline{x},0)\sigma + \cdots \tag{3.36}$$

Let

$$\overline{g}_x = g_x(\overline{x},0),\ \overline{g}_\sigma = g_\sigma(\overline{x},0) \text{ and} \tag{3.37}$$

$$\overline{h}_x = h_x(\overline{x},0),\ \overline{h}_\sigma = h_\sigma(\overline{x},0). \tag{3.38}$$

The goal is to find the $(n_y \times n_x)$ matrix $\overline{g}_x$, the $(n_y \times 1)$ vector $\overline{g}_\sigma$, the $(n_x \times n_x)$ matrix $\overline{h}_x$, and the $(n_x \times 1)$ vector $\overline{h}_\sigma$. The total of unknowns is, thus,

$$(n_x + n_y) \times (n_x + 1) = n \times (n_x + 1).$$

We solve for these unknowns by imposing

$$F_x(\overline{x},0) = 0, \tag{3.39}$$

which gives us $n \times n_x$ equations and

$$F_\sigma(\overline{x},0) = 0. \tag{3.40}$$

which gives us n equations.

To help with the exposition we introduce some notation. In particular, we denote the derivative of the i-th element of f with respect to the k-th element of z, for $z \in \{x,y\}$, with

$$\frac{\partial f^i}{\partial z^k} = [f_z]^i_k. \tag{3.41}$$

To understand the notation consider functions $v = v(x)$ and $w = w(x)$, which map R^{n_1} into R^{n_1} and a function $D(v,w)$ which maps R^{2*n_1} into R^{n_2}. Now consider the function $D(v(x),w(x))$. The derivative of the i-th element of D with respect to the j-th element of x is equal to

$$\frac{\partial D^i(v(x),w(x))}{\partial x^j} = \sum_{k_v=1}^{n_1} \frac{\partial f^i}{\partial v^{k_v}} \frac{\partial v^{k_v}}{\partial x^j} + \sum_{k_w=1}^{n_1} \frac{\partial f^i}{\partial w^{k_w}} \frac{\partial w^{k_w}}{\partial x^j}. \tag{3.42}$$

We will denote this by

$$\frac{\partial D^i(v(x),w(x))}{\partial x^j} = [f_v]^i_{k_v}\,[v_x]^{k_v}_j + [f_w]^i_{k_w}\,[w_x]^{k_w}_j\,. \tag{3.43}$$

That is, the index k showing up as a subscript and a superscript in adjacent terms indicates the summation. Moreover, the subscript of k indicates over how many elements the summation is. That is, k_v implies summing from $k_v = 1$ up to the number of elements of v.

We can use the same convenient notation if there are no derivatives involved. If η is an $(n_x \times n_\varepsilon)$ matrix and ε is an $(n_\varepsilon \times 1)$ vector then

$$\sum_{k_\varepsilon=1}^{n_\varepsilon} \eta_{\_i,k} \varepsilon_{\_k} = [\eta]^i_{k_\varepsilon} [\varepsilon']^{k_\varepsilon}, \tag{3.44}$$

where $\eta_{\_i,k}$ is the (i,k) element of η and $\varepsilon_{\_k}$ the k-th element of ε.

Using this notation to calculate the $n \times n_x$ derivatives of F with respect to x, we get

$$[F_x(\overline{x},0)]^i_j = \begin{array}{c} \left[\overline{f}_x\right]^i_j + \\ \left[\overline{f}_{x'}\right]^i_{k_h} \left[\overline{h}_x\right]^{k_h}_j + \\ \left[\overline{f}_y\right]^i_{k_g} [\overline{g}_x]^{k_g}_j + \\ \left[\overline{f}_{y'}\right]^i_{k_g} [\overline{g}_x]^{k_g}_{k_h} \left[\overline{h}_x\right]^{k_h}_j \end{array} = 0. \tag{3.45}$$

An upper bar over the function f indicates that the function is evaluated at the steady state values of x and y and at $\sigma = 0$.[2] Note that the $\overline{h}_x$ and $\overline{g}_x$ terms are multiplied. This is, thus, a second-order system of equations in the $n_x \times n_x$ values of $\overline{h}_x$ and the $n_y \times n_x$ values of $\overline{g}_x$. Although the notation is new, this part of the perturbation routine is identical to what has been done for many years by linearizing the first-order conditions. But the perturbation procedure adds the h_σ and g_σ coefficients. Those are solved from

$$[F_\sigma(\overline{x},0)]^i_j = \begin{array}{c} \mathrm{E}_t\left\{ \left[\overline{f}_{x'}\right]^i_{k_h} \left[\overline{h}_\sigma\right]^{k_h} + \left[\overline{f}_{x'}\right]^i_{k_h} [\eta]^{k_h}_{k_\varepsilon} [\varepsilon']^{k_\varepsilon} \right\} + \\ \mathrm{E}_t\left\{ \left[\overline{f}_y\right]^i_{k_g} [\overline{g}_\sigma]^{k_g} \right\} + \\ \mathrm{E}_t\left\{ \left[\overline{f}_{y'}\right]^i_{k_g} [\overline{g}_x]^{k_g}_{k_h} \left[\overline{h}_\sigma\right]^{k_h} + \left[\overline{f}_{y'}\right]^i_{k_g} [\overline{g}_x]^{k_g}_{k_h} [\eta]^{k_h}_{k_\varepsilon} [\varepsilon']^{k_\varepsilon} \right\} \\ \mathrm{E}_t\left\{ \left[\overline{f}_{y'}\right]^i_{k_g} [\overline{g}_\sigma]^{k_g} \right\} \end{array} = 0. \tag{3.46}$$

This gives us n equations to solve for $\overline{g}_\sigma$ and $\overline{h}_\sigma$. Below we will show that these coefficients are zero so that first-order perturbation will imply the same answer as "old-fashioned" linearization of the first-order conditions.

3.5 How does uncertainty matter?

The two important contributions of the perturbation procedure are that it allows for higher-order approximation and that it is explicit about the

[2] In other papers in the literature it is implicit that the function is evaluated at the steady state values and the upper bar is not used.

role of uncertainty. The question arises how important uncertainty is and whether it matters in lower-order approximations. Below we will discuss the very important results from Schmitt-Grohé and Uribe (2004). These results are actually quite straightforward to derive using the notation developed above, but the notation also hides a bit what is actually going on. So let's go back to the standard growth model and work out the equations in this simpler setup. In the standard growth model we would have

$$
\begin{aligned}
F(x,\sigma) &= \mathrm{E}_t f(k,\theta,k',\theta',c,c') \\
&= \mathrm{E}_t f\begin{pmatrix} k, \\ \theta \\ h(k,\theta,\sigma), \\ \rho\theta+\sigma\varepsilon', \\ g(k,\theta,\sigma), \\ g(h(k,\theta,\sigma),\rho\theta+\sigma\varepsilon',\sigma) \end{pmatrix},
\end{aligned}
$$

where f represents the budget constraint, the Euler equation, and the law of motion for θ. It thus maps R^6 into a (3×1) vector.

3.5.1 Uncertainty and first-order perturbation

To obtain the coefficients $\overline{g}_\sigma$ and $\overline{h}_\sigma$ (which are scalars in this case) we use

$$F_\sigma(\overline{x},0)=0. \tag{3.47}$$

$$
F_\sigma(x,0)=\mathrm{E}_t\begin{pmatrix} f_{k'}(s)h_\sigma(k,\theta,\sigma)+ \\ f_{\theta'}(s)\varepsilon'+ \\ f_c(s)g_\sigma(k,\theta,\sigma)+ \\ f_{c'}(s)\begin{pmatrix} g_k(k',\theta',\sigma)h_\sigma(k,\theta,\sigma)+ \\ g_\theta(k',\theta',\sigma)\varepsilon'+g_\sigma(k',\theta',\sigma) \end{pmatrix} \end{pmatrix} \tag{3.48}
$$

Here s denotes the arguments of f, that is, $s=[k,\theta,k',\theta',c,c']$. Evaluating this expression at $\overline{x}=0$ and calculating the expectation gives

$$F_\sigma(\overline{x},0)=\left(\overline{f}_{k'}+\overline{f}_{c'}\overline{g}_k\right)\overline{h}_\sigma+\left(\overline{f}_c+\overline{f}_{c'}\right)\overline{g}_\sigma=0 \tag{3.49}$$

Note that this system gives us three equations, since f consists of three elements, in the two unknowns $\overline{g}_\sigma$ and $\overline{h}_\sigma$. But note that the equation corresponding to the law of motion of productivity gives $0=0$ so we are left with two equations in two unknowns.[3] In particular, if we let f^{bc}

[3] Note that $\bar{g}_k$ is known. It is solved from $F_x(\bar{x},0)=0$.

denote the element of f corresponding to the budget constraint and f^{eu} the element of f corresponding the Euler equation we get

$$\begin{bmatrix} \overline{f}^{bc}_{k'} + \overline{f}^{bc}_{c'}\overline{g}_k & \overline{f}^{bc}_{c} + \overline{f}^{bc}_{c'} \\ \overline{f}^{eu}_{k'} + \overline{f}^{eu}_{c'}\overline{g}_k & \overline{f}^{eu}_{c} + \overline{f}^{eu}_{c'} \end{bmatrix} \begin{bmatrix} \overline{h}_\sigma \\ \overline{g}_\sigma \end{bmatrix} = 0 \tag{3.50}$$

So

$$\overline{g}_\sigma = \overline{h}_\sigma = 0 \tag{3.51}$$

and if there is no singularity in the system then this would be the unique solution. This, of course, corresponds to the certainty equivalence that one also obtains if linear policy rules are obtained by using the LQ procedure. Although we only show this result in a simple model, Schmitt-Grohé and Uribe (2004) prove that this result holds in more general frameworks.

3.5.2 Uncertainty and second-order perturbation

The second-order Taylor expansions of the policy functions are

$$\begin{aligned} h(k,\theta,\sigma) &= \overline{k} + \overline{h}_k(k-\overline{k}) + \overline{h}_\theta(\theta-\overline{\theta}) + \overline{h}_\sigma\sigma + 1/2(\\ &\quad +\overline{h}_{kk}(k-\overline{k})^2 + 2\overline{h}_{k\theta}(k-\overline{k})(\theta-\overline{\theta}) + 2\overline{h}_{k\sigma}(k-\overline{k})\sigma \\ &\quad +\overline{h}_{\theta\theta}(\theta-\overline{\theta})^2 + 2\overline{h}_{\theta\sigma}(\theta-\overline{\theta})\sigma + \overline{h}_{\sigma\sigma}\sigma^2) \;\text{ and} \end{aligned} \tag{3.52}$$

$$\begin{aligned} g(k,\theta,\sigma) &= \overline{c} + \overline{g}_k(k-\overline{k}) + \overline{g}_\theta(\theta-\overline{\theta}) + \overline{g}_\sigma\sigma + 1/2(\\ &\quad +\overline{g}_{kk}(k-\overline{k})^2 + 2\overline{g}_{k\theta}(k-\overline{k})(\theta-\overline{\theta}) + 2\overline{g}_{k\sigma}(k-\overline{k})\sigma \\ &\quad +\overline{g}_{\theta\theta}(\theta-\overline{\theta})^2 + 2\overline{g}_{\theta\sigma}(\theta-\overline{\theta})\sigma + \overline{g}_{\sigma\sigma}\sigma^2) \;\text{ and} \end{aligned} \tag{3.53}$$

From the discussion above we know that $\overline{g}_\sigma$ and $\overline{h}_\sigma$ are equal to zero. We will now show that $\overline{g}_{k\sigma}$, $\overline{g}_{\theta\sigma}$, $\overline{h}_{k\sigma}$, and $\overline{h}_{\theta\sigma}$ are equal to zero too. That is, the value of σ only shows up in the constant of the policy rules. Consider, for example, $\overline{g}_{k\sigma}$ and $\overline{h}_{k\sigma}$. These are solved from

$$F_{k\sigma}(\overline{x},0) = 0. \tag{3.54}$$

We get the derivative by differentiating the expression in Equation (3.48). This gives

$$F_{\sigma k}(x,\sigma) = \mathrm{E}_t \left(\begin{array}{c} (f_{k'k} + f_{k'k'}h_k + f_{k'c}g_k + f_{k'c'}g_k h_k)h_\sigma + \\ f_{k'}h_{\sigma k} + \\ (f_{\theta' k} + f_{\theta' k'}h_k + f_{\theta' c}g_k + f_{\theta' c'}g_k h_k)\varepsilon' + \\ (f_{ck} + f_{ck'}h_k + f_{cc}g_k + f_{cc'}g_k h_k)g_\sigma + \\ f_c g_{\sigma k} + \\ \left(\begin{array}{c} (f_{c'k} + f_{c'k'}h_k + f_{c'c}g_k + f_{c'c'}g_k h_k) \times \\ (g_k h_\sigma + g_\theta \varepsilon' + g_\sigma) \end{array} \right) + \\ f_{c'}(g_{kk}h_k h_\sigma + g_k h_{\sigma k}) \\ f_{c'}g_{\theta k}h_k\varepsilon' + \\ f_{c'}g_{\sigma k}h_k \end{array} \right). \tag{3.55}$$

Note that the arguments of the functions are suppressed. Here we stop at second order. That is we are not going to differentiate further and suppressing arguments doesn't matter. But if you do want to go on and obtain higher-order terms, it is better not to do so. That is, the above notation doesn't make clear whether g_k stands for $g_k(k,\theta,\sigma)$ or $g_k(k',\theta',\sigma) = g_k(h(k,\theta,\sigma),\theta',\sigma)$ and of course this difference is important if you differentiate.

Evaluating the last expression at the steady state we get

$$\begin{aligned} F_{\sigma k}(\overline{x},0) &= \overline{f}_{k'}\overline{h}_{\sigma k} + \overline{f}_c\overline{g}_{\sigma k} + \overline{f}_{c'}(\overline{g}_k\overline{h}_{\sigma k}) + \overline{f}_{c'}\overline{g}_{\sigma k}\overline{h}_k & (3.56) \\ &= \left(\overline{f}_{k'} + \overline{f}_{c'}\overline{g}_k\right)\overline{h}_{\sigma k} + \left(\overline{f}_c + \overline{f}_{c'}\right)\overline{g}_{\sigma k}\overline{h}_k = 0 & (3.57) \end{aligned}$$

Again we have two independent equations (together with $0 = 0$) in two unknowns $\overline{g}_{\sigma k}$ and $\overline{h}_{\sigma k}$. The solution is $\overline{g}_{\sigma k} = \overline{h}_{\sigma k} = 0$ and unless there is a singularity this is the unique solution.

So only the constants $\overline{g}_{\sigma\sigma}$ and $\overline{h}_{\sigma\sigma}$ are affected by the amount of uncertainty. Do not underestimate the importance of this. A different value for the constant term in a policy rule implies that the system will operate in a different part of the state space. For example, the higher coefficients can capture a buffer stock motive that induces agents to have on average higher capital stocks. Now if the second-order terms of capital are not equal to zero then being in a different part of the state space also implies that the response of the system depends on where you are. Consequently, different values for $\overline{g}_{\sigma\sigma}$ and $\overline{h}_{\sigma\sigma}$ can indirectly also affect how sensitive the economy is to shocks.

3.6 Why not all ways to do LQ approximations are correct

A linear quadratic dynamic programming problem has a quadratic objective and linear constraints.[4] This problem, the optimal linear regulator problem is studied extensively and can be solved without numerical error. Suppose that one has a problem for which the problem is not quadratic and the constraints not linear. One might be tempted to take a quadratic approximation of the objective function and a linear approximation to the constraints and then solve the optimal linear regulator problem. This turns out not to be the right way to implement the LQ procedure in the sense that the linear solution does not in general correspond to the first-order Taylor expansion of the true solution. We will document this using a simple example. In particular, we will first give the set of equations that determine the first-order solution of the perturbation procedure, which by construction gives the first-order Taylor expansion of the true policy function. This makes clear that the correct first-order Taylor expansion also depends on second-order terms of the constraint. Benigno and Woodford (2006) refer to this way to take LQ approximations as the "naive LQ approximation". Next we use the example to discuss a modified LQ procedure that is correct.

We focus on the following model.

$$\begin{gathered}\max_{x,y}\min_{\lambda} f(x,y,a)+\lambda(b-g(x,y,a)) \\ \text{s.t.}\ \ \lambda\geq 0\end{gathered}$$

Here f and g are scalar functions and x, y, and a are scalars as well. The simple structure will be helpful in clarifying the points made but the arguments are much more general. The first-order conditions can be written as

$$\begin{aligned} f_x(x,y,a)-\lambda g_x(x,y,a) &= 0, && (3.58)\\ f_y(x,y,a)-\lambda g_y(x,y,a) &= 0, \text{ and} && (3.59)\\ g(x,y,a) &= b. && (3.60)\end{aligned}$$

The solutions to this system of equations are

$$\begin{aligned} x &= h^x(a), && (3.61)\\ y &= h^y(a), \text{ and} && (3.62)\\ \lambda &= h^\lambda(a). && (3.63)\end{aligned}$$

[4] One can only deal with linear constaints since these are multiplied by the Lagrange multipliers in the Lagrangian making the constraint term also of second order.

Using this, we can write the first-order conditions as

$$f_x(h^x(a), h^y(a), a) - \lambda(a) g_x(h^x(a), h^y(a), a) = 0, \tag{3.64}$$
$$f_y(h^x(a), h^y(a), a) - \lambda(a) g_y(h^x(a), h^y(a), a) = 0, \text{ and} \tag{3.65}$$
$$g(h^x(a), h^y(a), a) = b. \tag{3.66}$$

The Taylor expansions of the policy functions are given by

$$h^x(a) = \overline{h}^x + \overline{h}^x_a(a-\overline{a}) + \overline{h}^x_{aa}(a-\overline{a})^2 + \cdots, \tag{3.67}$$
$$h^y(a) = \overline{h}^y + \overline{h}^y_a(a-\overline{a}) + \overline{h}^y_{aa}(a-\overline{a})^2 + \cdots, \text{ and} \tag{3.68}$$
$$h^\lambda(a) = \overline{h}^\lambda + \overline{h}^\lambda_a(a-\overline{a}) + \overline{h}^\lambda_{aa}(a-\overline{a})^2 + \cdots \tag{3.69}$$

The first-order derivatives evaluated at $a = \overline{a}$ are $\overline{h}^x_a$, $\overline{h}^y_a$, and $\overline{h}^\lambda_a$. Differentiating the first-order conditions and evaluating them at $a = \overline{a}$ gives

$$\overline{f}_{xx}\overline{h}^x_a + \overline{f}_{xy}\overline{h}^y_a + \overline{f}_{xa} - \overline{\lambda}(\overline{g}_{xx}\overline{h}^x_a + \overline{g}_{xy}\overline{h}^y_a + \overline{g}_{xa}) - \overline{g}_x\overline{h}^\lambda_a = 0, \tag{3.70}$$
$$\overline{f}_{yx}\overline{h}^x_a + \overline{f}_{yy}\overline{h}^y_a + \overline{f}_{ya} - \overline{\lambda}(\overline{g}_{yx}\overline{h}^x_a + \overline{g}_{yy}\overline{h}^y_a + \overline{g}_{ya}) - \overline{g}_y\overline{h}^\lambda_a = 0, \tag{3.71}$$
$$\overline{g}_x\overline{h}^x_a + \overline{g}_y\overline{h}^y_a + \overline{g}_a = 0. \tag{3.72}$$

This can be written as

$$\begin{bmatrix} \overline{f}_{xx} - \overline{\lambda}\overline{g}_{xx} & \overline{f}_{xy} - \overline{\lambda}\overline{g}_{xy} & -\overline{g}_x \\ \overline{f}_{yx} - \overline{\lambda}\overline{g}_{yx} & \overline{f}_{yy} - \overline{\lambda}\overline{g}_{yy} & -\overline{g}_y \\ -\overline{g}_x & -\overline{g}_y & 0 \end{bmatrix} \begin{bmatrix} \overline{h}^x_a \\ \overline{h}^y_a \\ \overline{h}^\lambda_a \end{bmatrix} = \begin{bmatrix} -\overline{f}_{xa} + \overline{\lambda}\overline{g}_{xa} \\ -\overline{f}_{ya} + \overline{\lambda}\overline{g}_{ya} \\ -\overline{g}_a \end{bmatrix} \tag{3.73}$$

With this system we can solve for the first-order perturbation terms. Note that they depend on the second-order properties of the constraints ($\overline{g}_{xx}$, $\overline{g}_{xy}$, etc.). These would not show up when one linearizes the constraints.

3.7 Correct LQ procedure

There are different ways in which one can deal with the problem encountered above. One could try to get rid of the constraints (or make them linear) by substituting out or redefining variables. This is not always possible. Here we discuss a general LQ procedure for which the linear solution does correspond to the first-order Taylor expansion.[5] That is, it corresponds

[5] This part of the note is based on Benigno and Woodford (2006) and has benefitted from comments by PierPaolo Benigno and Michael Woodford. Also see Altissimo, Curdia, and Rodrıguez-Palenzuela (2005) and Debortoli and Nunes (2006).

with the solution procedure implied by Equation (3.73). The standard formulation of the LQ approximation can be written as

$$\max_{x,y}\min_{\lambda} \begin{array}{c} \overline{f}_x\widetilde{x}+\overline{f}_y\widetilde{y}+\overline{f}_a\widetilde{a} \\ +\frac{1}{2}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}'\begin{bmatrix}\overline{f}_{xx} & \overline{f}_{xy} & \overline{f}_{xa}\\ \overline{f}_{yx} & \overline{f}_{yy} & \overline{f}_{ya}\\ \overline{f}_{ax} & \overline{f}_{ay} & \overline{f}_{aa}\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix} \\ +\lambda\left[-\overline{g}_x\widetilde{x}-\overline{g}_y\widetilde{y}-\overline{g}_a\widetilde{a}\right]\end{array} \tag{3.74}$$

Here $\widetilde{z}=z-\overline{z}$.Now consider a second-order approximation of the constraint

$$0\approx \begin{array}{c} \overline{g}_x\widetilde{x}+\overline{g}_y\widetilde{y}+\overline{g}_a\widetilde{a} \\ +\frac{1}{2}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}'\begin{bmatrix}\overline{g}_{xx} & \overline{g}_{xy} & \overline{g}_{xa}\\ \overline{g}_{yx} & \overline{g}_{yy} & \overline{g}_{ya}\\ \overline{g}_{ax} & \overline{g}_{ay} & \overline{g}_{aa}\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}\end{array} \tag{3.75}$$

Next we multiply both sides of this expression by $\overline{\lambda}$ and use the first-order conditions at the steady state values, i.e., at $a=\overline{a}$. This gives

$$\begin{array}{c} \overline{f}_x\widetilde{x}+\overline{f}_y\widetilde{y}+\overline{\lambda}\overline{g}_a\widetilde{a} \\ +\frac{1}{2}\overline{\lambda}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}'\begin{bmatrix}\overline{g}_{xx} & \overline{g}_{xy} & \overline{g}_{xa}\\ \overline{g}_{yx} & \overline{g}_{yy} & \overline{g}_{ya}\\ \overline{g}_{ax} & \overline{g}_{ay} & \overline{g}_{aa}\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}\end{array} \tag{3.76}$$

Subtracting this expression from the Lagrangian (and ignoring constant terms) gives

$$\max_{x,y}\min_{\lambda}\frac{1}{2}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix}'\begin{bmatrix}\overline{f}_{xx}-\overline{\lambda}\overline{g}_{xx} & \overline{f}_{xy}-\overline{\lambda}\overline{g}_{xy} & \overline{f}_{xa}-\overline{\lambda}\overline{g}_{xy}\\ \overline{f}_{yx}-\overline{\lambda}\overline{g}_{yx} & \overline{f}_{yy}-\overline{\lambda}\overline{g}_{yy} & \overline{f}_{ya}-\overline{\lambda}\overline{g}_{ya}\\ \overline{f}_{ax}-\overline{\lambda}\overline{g}_{ax} & \overline{f}_{ay}-\overline{\lambda}\overline{g}_{ay} & \overline{f}_{aa}-\overline{\lambda}\overline{g}_{aa}\end{bmatrix}\begin{bmatrix}\widetilde{x}\\ \widetilde{y}\\ \widetilde{a}\end{bmatrix} \tag{3.77}$$

$$+\lambda\left[b-\overline{g}-\overline{g}_x\widetilde{x}-\overline{g}_y\widetilde{y}-\overline{g}_a\widetilde{a}\right]$$

By entering second-order properties of the constraint we have at least some chance of getting the correct first-order Taylor expansion. Also note that the first-order terms have disappeared from the objective function. This is already a convenient property. The linear solution that comes out of this is going to be at best the correct first-order Taylor expansion. Consequently, it is in general wrong in the second-order terms. But if you substitute such a policy function into the linear terms of the objective function then the objective function also has second-order mistakes. But the objective function is supposed to be the correct second-order approximation.

Anyway, we are really interested in knowing whether this leads to the correct first-order Taylor expansion. To see this calculate the first-order conditions of this problem. They are given by

$$\begin{bmatrix} \overline{f}_{xx} - \overline{\lambda}\overline{g}_{xx} & \overline{f}_{xy} - \overline{\lambda}\overline{g}_{xy} & -\overline{g}_x \\ \overline{f}_{yx} - \overline{\lambda}\overline{g}_{yx} & \overline{f}_{yy} - \overline{\lambda}\overline{g}_{yy} & -\overline{g}_y \\ -\overline{g}_x & -\overline{g}_y & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} -\overline{f}_{xa} + \overline{\lambda}\overline{g}_{xa} \\ -\overline{f}_{ya} + \overline{\lambda}\overline{g}_{ya} \\ -\overline{g}_a \end{bmatrix} \widetilde{a}. \tag{3.78}$$

But this system corresponds exactly to Equation (3.73). [6]

Modification of the objective function and change in Lagrange multiplier

The procedure outlined above boils down to subtracting the second-order formulation of the constraint given in Equation (3.76) from the original objective function given in (3.74). Thus, whenever the constraints hold, this does not entail any change in the objective function since one would simply be adding zero to the objective function. This is not true when the constraints are not satisfied, so we are indeed changing the objective function.

This shift in the objective function does have an effect on the Lagrange multiplier. In fact, Equation (3.78) makes clear that when $\widetilde{a} = 0$, that the value of the Lagrange multiplier is equal to zero.[7] But we have also seen that this shift does *not* have an effect on the first-order term that relates changes in $\widetilde{a}$ to changes in λ. An easy way to "undo" the effect of the shift of the objective function on the Lagrange multiplier is to use the following modified LQ specification.

$$\max_{x,y} \min_{\lambda} \frac{1}{2} \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{a} \end{bmatrix}' \begin{bmatrix} \overline{f}_{xx} - \overline{\lambda}\overline{g}_{xx} & \overline{f}_{xy} - \overline{\lambda}\overline{g}_{xy} & \overline{f}_{xa} - \overline{\lambda}\overline{g}_{xy} \\ \overline{f}_{yx} - \overline{\lambda}\overline{g}_{yx} & \overline{f}_{yy} - \overline{\lambda}\overline{g}_{yy} & \overline{f}_{ya} - \overline{\lambda}\overline{g}_{ya} \\ \overline{f}_{ax} - \overline{\lambda}\overline{g}_{ax} & \overline{f}_{ay} - \overline{\lambda}\overline{g}_{ay} & \overline{f}_{aa} - \overline{\lambda}\overline{g}_{aa} \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{a} \end{bmatrix} \tag{3.79}$$

$$+\widetilde{\lambda}\left[b - \overline{g} - \overline{g}_x\widetilde{x} - \overline{g}_y\widetilde{y} - \overline{g}_a\widetilde{a}\right].$$

That is, we replace λ by $\widetilde{\lambda} = \lambda - \overline{\lambda}$. Note that this is not a replacement of the Lagrange multiplier term by its first-order approximation. This would give $\lambda = \widetilde{\lambda} + \overline{\lambda}$. Obviously, replacing λ by $\widetilde{\lambda}$ is just a change in notation and the value for $\widetilde{\lambda}$ obtained with (3.79) will be identical to the value for λ obtained with (3.77). Thus, by using (3.79) we will now get that $\widetilde{\lambda} = 0$ when $\widetilde{a} = 0$. But this means that if we use (3.79) we get $\lambda = \overline{\lambda}$ when $\widetilde{a} = 0$, which is the desired outcome.

[6] Note that the modification of the objective function results in a zero value of the Lagrange multiplier when $a = 0$ even though in the original problem $\overline{\lambda} > 0$. Note that by construction the constraint does still hold with equality when $a = 0$. If one is interested in getting the right value of λ that is close to that of the original problem one can replace λ by $(\lambda - \overline{\lambda})$ in (3.77).

[7] Note that although the Lagrange multiplier is zero, the constraint is still satisfied at $\widetilde{a} = 0$, it is just not binding.

In other words, the effect of the shift in the objective function due to adding the second-order approximation of the constraint is undone by a shift in the Lagrange multiplier.

Exercise 1 *Consider the following model*

$$\begin{aligned} E\left[f(k', e')\right] &= 0 \\ e' &= \sigma\varepsilon', \end{aligned}$$

where ε' is a random variable with the following properties

$$\begin{aligned} E\left[\varepsilon'\right] &= 0 \\ E\left[(\varepsilon')^2\right] &= \zeta_2 \\ E\left[(\varepsilon')^3\right] &= \zeta_3 \neq 0 \end{aligned}$$

Note that there are no state variables in this model. The solution is, thus, simply a constant, but this constant choice does depend on the model parameters, σ, ζ_2, and ζ_3. The idea of perturbation is to find a solution by solving for the coefficients of the Taylor expansion

$$k' = g(\sigma).$$

Show that the first-order, second-order, and third-order coefficient are equal to

$$\begin{aligned} \bar{g}_\sigma &= -\frac{E\left[\varepsilon' f_{e'}(\bar{k}, 0)\right]}{f_k(\bar{k}, 0)} = 0, \\ \bar{g}_{\sigma^2} &= -\frac{f_{(e')^2}(\bar{k}, 0) E\left[(\varepsilon')^2\right]}{f_k(\bar{k}, 0)} = -\frac{f_{(e')^2}(\bar{k}, 0)\zeta_2}{f_k(\bar{k}, 0)}, \text{ and} \\ \bar{g}_{\sigma^3} &= \frac{f_{(e')^3}(\bar{k}, 0) E\left[(\varepsilon')^3\right]}{f_k(\bar{k}, 0)} = -\frac{f_{(e')^3}(\bar{k}, 0)\zeta_3}{f_k(\bar{k}, 0)}. \end{aligned}$$

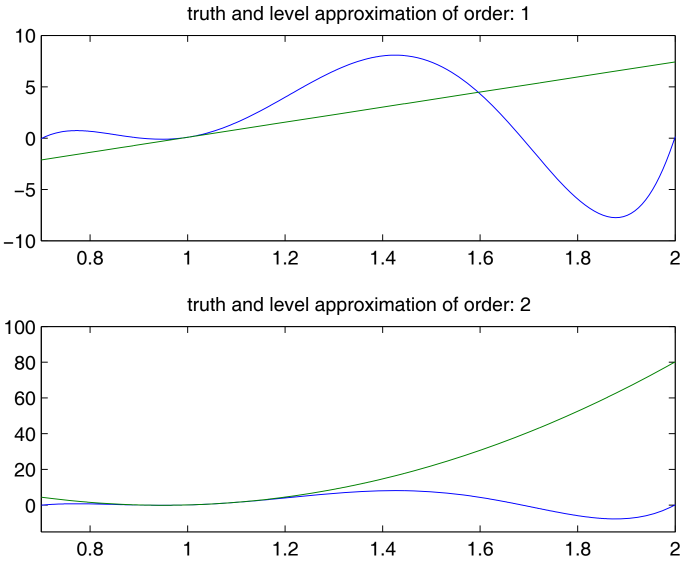

FIGURE 3.1. Level approximations

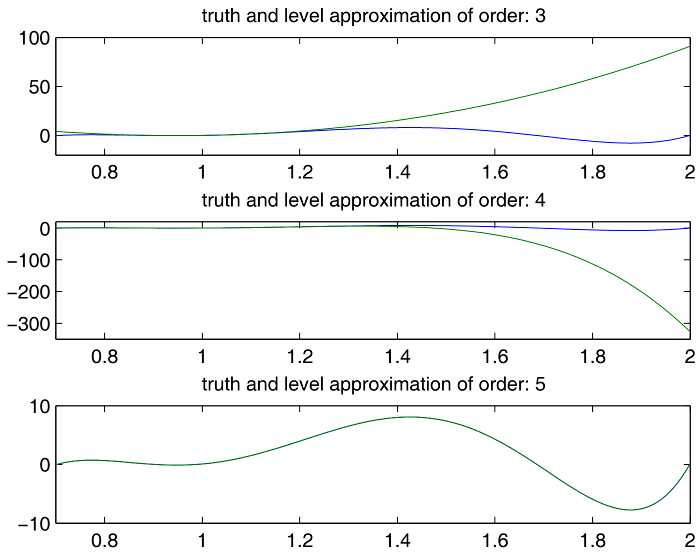

FIGURE 3.2. Level approximations continued

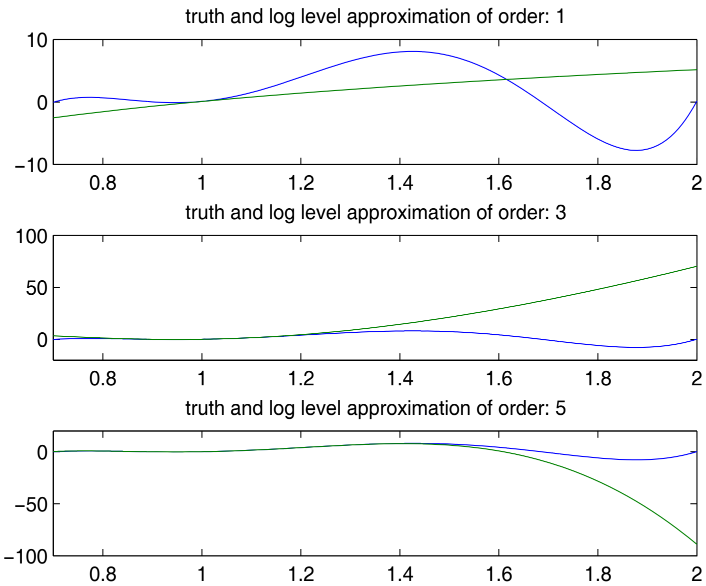

FIGURE 3.3. Log level approximations

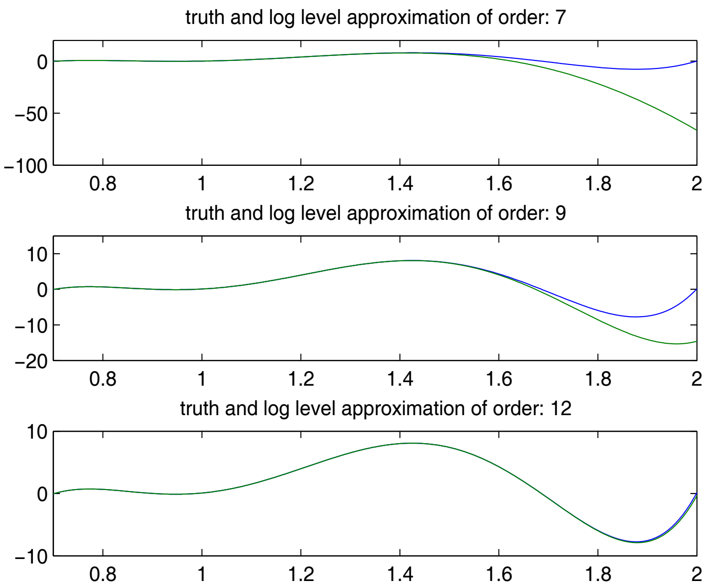

FIGURE 3.4. Log level approximations continued