Solving Models with Heterogeneous Agents Xpa algorithm

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Individual agent

- Subject to employment shocks
  - $\varepsilon_{i,t} \in \{0, 1\}$ or $\varepsilon_{i,t} \in \{u, e\}$
- Incomplete markets
  - only way to save is through holding capital

Alternatives to inequality constraint

- Penalty function in the utility function

$$u(c_{i,t}, k_{i,t}) = \ln(c_{i,t}) - P(k_{i,t+1})$$

- Assumptions about $P(k_{i,t})$

standard inequality: $k_{i,t+1} \geq 0$ differentiable alternative

$$P(k_{i,t+1}) = \begin{cases} \infty \text{ if } k_{i,t+1} < 0 \\ 0 \text{ if } k_{i,t+1} \geq 0 \end{cases} \qquad p(k_{i,t+1}) = \frac{\partial P(k_{i,t+1})}{\partial k_{i,t+1}} \leq 0$$

Alternatives to inequality constraint

Alternative to $P(\cdot)$ in utility function

- Individual interest rate depends on amount invested

$$r_{i,t} = r_t - P(k_{i,t+1}) \text{ with } \frac{\partial P(k_{i,t+1})}{\partial k_{i,t+1}} > 0$$

- Advantage: nicer economic story
- Disadvantage:
  - you have to take a stand on what happens with the resources (thrown away? intermediary?)
  - both Euler and budget constraint are affected

Laws of motion

- aggregate productivity, z_t, can take on two values
- employment status, ε_t, can take on two values
- probability of being (un)employed depends on z_t
- transition probabilities are such that unemployment rate only depends on current z_t
- $\Longrightarrow u_t = u(z_t)$ with $u_b = u(1 - \Delta_z) > u_g - u(1 + \Delta_z)$.

Individual agent

$$\max_{\{c_{i,t},k_{i,t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^{t}\left(\ln(c_{i,t})-P(k_{i,t+1})\right)$$

s.t.

$$c_{i,t}+k_{i,t+1}=r_{t}k_{i,t}+(1-\tau_{t})w_{t}\bar{l}\varepsilon_{i,t}+\mu w_{t}(1-\varepsilon_{i,t})+(1-\delta)k_{i,t}$$

for **given** processes of r_t and w_t, this is a relatively simple problem

Individual agent: Euler equation

$$\frac{1}{c_{i,t}} \;=\; p(k_{i,t+1}) + \mathsf{E}_t\left[\frac{\beta(r_{t+1}+1-\delta)}{c_{i,t+1}}\right]$$

cost of $k_{i,t+1}$ $\uparrow$ $=$ reduction in penalty $+$ usual term

Firm problem

$$
\begin{aligned}
r_t &= z_t \alpha \left(\frac{K_t}{\bar{l}(1 - u(z_t))} \right)^{\alpha - 1} \\
w_t &= z_t (1 - \alpha) \left(\frac{K_t}{\bar{l}(1 - u(z_t))} \right)^{\alpha}
\end{aligned}
$$

Government

$$
\begin{aligned}
\tau_t w_t \bar{l}(1-u(z_t)) &= \mu w_t u(z_t) \\
\tau_t &= \frac{\mu u(z_t)}{\bar{l}(1-u(z_t))}
\end{aligned}
$$

Aggregate variables agents care about

- r_t and w_t
- They only depend on aggregate capital stock and z_t
- !!! This is not true in general for equilibrium prices
- Agents are interested in all information that forecasts K_t
- In principle that is the complete cross-sectional distribution of employment status and capital levels

Equilibrium - first part

- Individual policy functions solving the agent's max problem
- A wage and a rental rate given by equations above.

Equilibrium - second part

- A transition law for the cross-sectional distribution of capital, consistent with the individual policy function.
  - $f_t =$ beginning-of-period cross-sectional distribution of capital and the employment status *after* the employment status has been realized.

$$f_{t+1} = \Upsilon(z_{t+1}, z_t, f_t)$$

  - z_{t+1} does not affect the period t cross-sectional distribution of capital
  - z_{t+1} does affect the *joint* cross-sectional distribution of capital and employment status

Overview

1. What if individual policy rules are linear in *levels*?
2. What if individual policy rules are polynomials in *levels*
3. What if individual policy rules are *not* polynomials in the levels
4. Topics:
    - What if there are non-differentiabilities
    - Economy with bonds
        - the price of a bond—unlike rental rate—is not a simple function of K_t

Linear policy rule for individual problem

- Suppose policy rules are given and equal to:

$$\begin{aligned} \text{if } \varepsilon &= u: \; k'_u = \Psi_{u,0}(s) + \Psi_{u,1}(s)k \\ \text{if } \varepsilon &= e: \; k'_e = \Psi_{e,0}(s) + \Psi_{e,1}(s)k \end{aligned}$$

- linear in k, completely general in all other dimensions

Linear policy rule for individual problem

- s is the set of aggregate state variables and consist for sure of z, K_u, K_e

- Xpa determines "endogenously" whether other elements should be added to this list

Linear policy rule for individual problem

- Policy rules:

$$\begin{aligned}\text{if } \varepsilon &= u: \; k'_u = \Psi_{u,0}(s) + \Psi_{u,1}(s)k \\ \text{if } \varepsilon &= e: \; k'_e = \Psi_{e,0}(s) + \Psi_{e,1}(s)k\end{aligned}$$

- Can we calculate K' from this?
- If the answer is yes, then we can calculate r' (given z')

Notation

End of this period

- $\widehat{K}_u$: *end-of-period* aggregate capital stock of unemployed
- $\widehat{K}_e$: *end-of-period* aggregate capital stock of employed

Beginning of the next period

- K'_u and K'_e: corresponding *beginning-of-period* equivalents

Transition laws

From end of this period to beginning of next period

- $K'_u \neq \widehat{K}_u$ and $K'_e \neq \widehat{K}_e$ because employment status changes
- Apply transition laws to go from $\widehat{K}_e, \widehat{K}_u$ to K'_e, K'_u

More notation

- $g_{\varepsilon\varepsilon' zz'}$: mass of agents with employment status ε now and ε' next period for given values of z and z'

$$g_{uuzz'} + g_{euzz'} + g_{eezz'} + g_{uezz'} = 1$$

From end to beginning-of-period moments

$$K'_u = \frac{g_{uuzz'}\widehat{K}_u + g_{euzz'}\widehat{K}_e}{g_{uuzz'} + g_{euzz'}}$$

$$K'_e = \frac{g_{uezz'}\widehat{K}_u + g_{eezz'}\widehat{K}_e}{g_{uezz'} + g_{eezz'}}$$

$$K' = u(z')K'_u + (1 - u(z'))K'_e$$

Given the formulas for $\widehat{K}_u$ and $\widehat{K}_e$ we can calculate these.

Simpler case (to learn method)

Suppose individual policy function are as follows:

$$\begin{aligned}\text{if } \varepsilon &= u: \; k'_u = \Psi_{u,0} + \Psi_{u,k}k + \Psi_{u,z}z + \Psi_{u,K_u}K_u + \Psi_{u,K_e}K_e \\ \text{if } \varepsilon &= e: \; k'_e = \Psi_{e,0} + \Psi_{e,k}k + \Psi_{e,z}z + \Psi_{e,K_u}K_u + \Psi_{e,K_e}K_e\end{aligned}$$

This immediately gives

$$\begin{gathered}\widehat{K}_u = \widehat{M}_u(1) = \int k'_u(\cdot)dF_u(k) \\ = (\Psi_{u,0} + \Psi_{u,z}z) + (\Psi_{u,k} + \Psi_{u,K_u})\,K_u + \Psi_{u,K_e}K_e, \\ \widehat{K}_e = \widehat{M}_e(1) = \int k'_e(\cdot)dF_e(k) \\ = (\Psi_{e,0} + \Psi_{e,z}z) + (\Psi_{e,k} + \Psi_{e,K_e})\,K_e + \Psi_{e,K_u}K_u,\end{gathered}$$

- Applying transition laws gives K'_u, K'_e, and K'

What about the state variables?

- Above, we assumed that s consists of $k, \varepsilon, z, K_u, K_e$ and possibly other characteristics of the distribution. But with linear policy rules, no other moments are needed.

- Intuition?

Simple way to solve for coefficients

1. Assume linear law of motion for K_u' and K_e'
2. Find linear approximation for k_u' and k_e'
3. Get new linear law of motion for K_u' and K_e' by *explicit aggregation* (and transition laws)
4. Iterate until convergence

More general linear policy rules

Policy rules:

$$\begin{aligned} \text{if } \varepsilon &= u: \; k'_u = \Psi_{u,0}(s) + \Psi_{u,1}(s)k \\ \text{if } \varepsilon &= e: \; k'_e = \Psi_{e,0}(s) + \Psi_{e,1}(s)k \end{aligned}$$

This immediately gives

$$\widehat{K}_u = \widehat{M}_u(1) = \int k'_u(k,\cdot)dF_u(k) = \Psi_{u,0}(s) + \Psi_{u,1}(s)M_u(1),$$
$$\widehat{K}_e = \widehat{M}_e(1) = \int k'_e(k,\cdot)dF_e(k) = \Psi_{e,0}(s) + \Psi_{e,1}(s)M_e(1),$$

- This law of motion could be non-linear in K_u and K_e!!!!
- Applying transition laws gives K'_u, K'_e, and K'

Progress so far

- Given linear individual policy rule:
  - we can get law of motion for aggregate variables
  - determine what the set of aggregate state variables are
- Haven't said anything yet on how to find individual policy rules
  - We'll answer that for somewhat more general case with quadratic individual policy rules

Second-order policy rule

Policy rules:

$$
\begin{aligned}
k'_u &= \Psi_{u,0}(s) + \sum_{i=1}^{2} \Psi_{u,i}(s) k^i \text{ and} \\
k'_e &= \Psi_{e,0}(s) + \sum_{i=1}^{2} \Psi_{e,i}(s) k^i,
\end{aligned}
$$

- quadratic in k, completely general in all other dimensions

Second-order policy rule

- s is the set of state variables and consist for sure of z, K_u, K_e
- Xpa determines "endogenously" whether other elements should be added to this list

Aggregate second-order policy

- Aggregation gives

$$\widehat{K}_u = \widehat{M}_u(1) = \Psi_{u,0}(s) + \textstyle\sum_{i=1}^{2} \Psi_{u,i}(s) M_u(i),$$
$$\widehat{K}_e = \widehat{M}_e(1) = \Psi_{e,0}(s) + \textstyle\sum_{i=1}^{2} \Psi_{e,i}(s) M_e(i),$$

- What are the state variables?
    - Not only $M_u(1)$ and $M_e(1)$, but also $M_u(2)$ and $M_e(2)$

Aggregate second-order policy

- $\Longrightarrow$ we need laws of motion for $\widehat{M}_u(2)$ and $\widehat{M}_e(2)$;
    - these together with transition laws would give laws of motion for $M'_u(2)$ and $M'_e(2)$

Laws of motion for second-order terms

- Moments are

$$\widehat{M}_\varepsilon(2) = \int \left(k'_\varepsilon(k,\cdot)\right)^2 dF_\varepsilon(k)$$

- Do I have a law of motion for $\left(k'_\varepsilon\right)^2$? Yes, of course, namely

$$\left(k'_\varepsilon\right)^2 = \begin{array}{c} \Psi^2_{\varepsilon,0}(s) + 2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,1}(s)k + \\ (2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,2} + (\Psi_{\varepsilon,1}(s))^2)k^2 \\ +2\Psi_{\varepsilon,1}(s)\Psi_{\varepsilon,2}(s)k^3 + \left(\Psi_{\varepsilon,2}(s)\right)^2 k^4. \end{array}$$

What is the problem?

Laws of motion for second-order terms

- Aggregating

$$\left(k'_{\varepsilon}\right)^2 = \begin{array}{c} \Psi^2_{\varepsilon,0}(s) + 2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,1}(s)k + \\ (2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,2} + (\Psi_{\varepsilon,1}(s))^2)k^2 \\ +2\Psi_{\varepsilon,1}(s)\Psi_{\varepsilon,2}(s)k^3 + (\Psi_{\varepsilon,2}(s))^2\, k^4. \end{array}$$

gives

$$\widehat{M}_{\varepsilon}(2) = \begin{array}{c} (\Psi_{\varepsilon,0}(s))^2 + 2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,1}(s)\widehat{M}_{\varepsilon}(1) \\ +(2\Psi_{\varepsilon,0}(s)\Psi_{\varepsilon,2} + (\Psi_{\varepsilon,1}(s))^2)\widehat{M}_{\varepsilon}(2) \\ +2\Psi_{\varepsilon,1}(s)\Psi_{\varepsilon,2}(s)\widehat{M}_{\varepsilon}(3) + (\Psi_{\varepsilon,2}(s))^2\,\widehat{M}_{\varepsilon}(4). \end{array}$$

Avoiding infinite-regress problem

- Define
$$y'_\varepsilon = \left(k'_\varepsilon\right)^2$$
Come up with a *separate* 2nd-order approximation for y'_ε:
$$y'_\varepsilon = \left(k'_\varepsilon\right)^2 \approx \Psi_{\varepsilon,(k')^2,0}(s) + \Psi_{\varepsilon,(k')^2,1}(s)k + \Psi_{\varepsilon,(k')^2,2}(s)k^2$$
- $\Psi_{\varepsilon,j}$ coeffs have no direct relationship to $\Psi_{\varepsilon,(k')^2,j}$ coeffs

Aggregation in second-order case

$$
\begin{aligned}
k'_{\varepsilon} &= \Psi_{\varepsilon,0}(s) + \sum_{i=1}^{2} \Psi_{\varepsilon,i}(s) k^{i} \text{ gives} \\
\widehat{K}_{\varepsilon} &= \widehat{M}_{\varepsilon}(1) = \Psi_{\varepsilon,0}(s) + \sum_{i=1}^{2} \Psi_{\varepsilon,i}(s) M_{\varepsilon}(i),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(k'_{\varepsilon}\right)^{2} &= \Psi_{\varepsilon,(k')^2,0}(s) + \sum_{i=1}^{2} \Psi_{\varepsilon,(k')^2,i}(s) k^{i} \text{ gives} \\
\widehat{M}_{\varepsilon}(2) &= \Psi_{\varepsilon,(k')^2,0}(s) + \sum_{i=1}^{2} \Psi_{\varepsilon,(k')^2,i}(s) M_{\varepsilon}(i),
\end{aligned}
$$

Basic formulation

- Policy rules:

$$k'_u = \Psi_{u,0}(s) + \sum_{i=1}^{I} \Psi_{u,i}(s) k^i \text{ and } k'_e = \Psi_{e,0}(s) + \sum_{i=1}^{I} \Psi_{e,i}(s) k^i,$$

- Aggregation gives

$$\begin{aligned} \widehat{K}_u = \widehat{M}_u(1) = \Psi_{u,0}(s) + \textstyle\sum_{i=1}^{I} \Psi_{u,i}(s) M_u(i), \\ \widehat{K}_e = \widehat{M}_e(1) = \Psi_{e,0}(s) + \textstyle\sum_{i=1}^{I} \Psi_{e,i}(s) M_e(i), \end{aligned}$$

- Get additional *separate* policy rules for each polynomial term in policy function

Full program

How to find the Ψ coefficients?

- Iterative perturbation procedures
- general projection method

Iterative perturbation solutions

1. Guess a law of motion for aggregate law of motion
2. Conditional on this solve for individual law of motion
3. Explicitly aggregate and update aggregate law of motions

Perturbation solution with discrete support

- Write law of motion for z_t and ε_t as

$$\begin{aligned} z_t &= \bar{z} + \rho_z z_{t-1} + e_{z,t} \quad \text{with } \mathsf{E}\left[e_{z,t}^2\right] = \sigma_z^2 \\ \varepsilon_{i,t} &= \bar{\varepsilon} + \rho_\varepsilon \varepsilon_{t-1} + e_{\varepsilon,t} \quad \text{with } \mathsf{E}\left[e_{\varepsilon,t}^2\right] = \sigma_\varepsilon^2 \end{aligned}$$

- $\rho_z, \rho_\varepsilon, \sigma_z^2, \sigma_\varepsilon^2$ are such that autocovariances and variances correspond to original process

General projection procedure

- Suppose a second-order solution is used:
  $s_{i,t} = [\varepsilon_{i,t}, k_{i,t}, z_t, M_{u,t}(1), M_{e,t}(1), M_{u,t}(2), M_{e,t}(2)]$
- Basic idea:
  - set up a grid
  - define error term at each grid point
  - define loss function
  - use minimization routine to find Ψ coefficients

General projection procedure

- Solve for $\psi_{k'}(s;\Psi)$ by making model equations "fit" on grid
- Notation: we have included $\varepsilon_{i,t}$ in s and $\psi_{k'}(s;\Psi)$ describes behavior of both employed and unemployed agent

Individual policy rules & projection methods

- $\{s_\kappa\}_{\kappa=1}^{\chi}$ the set of state variables with χ nodes
- $s_\kappa = \{z_\kappa, M_{u,\kappa}(1), M_{e,\kappa}(1), M_{u,\kappa}(2), M_{e,\kappa}(2)\}$
- κ indicates a grid point not a period

First-order condition

$$\left(\begin{array}{c} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -\psi_{k'}(s_\kappa;\Psi) \end{array}\right)^{-\nu}$$

$$= p(\psi_{k'}(s_\kappa;\Psi)) + \mathsf{E}\left[\begin{array}{c} \beta(r(z',K') + (1-\delta))\times \\ \left(\begin{array}{c} (r(z',K') + 1 - \delta)\psi_{k'}(s_\kappa;\Psi) \\ +(1-\tau(z'))w(z',K')\bar{l}\varepsilon' \\ +\mu w(z',K')(1-\varepsilon') \\ -\psi_{k'}(s';\Psi) \end{array}\right)^{-\nu} \end{array}\right]$$

Individual problem errors (usual part):

$$u_\kappa = -\left(\begin{array}{c} (r(z_\kappa, K_\kappa) + 1 - \delta)k_\kappa \\ +(1-\tau(z_\kappa))w(z_\kappa, K_\kappa)\bar{l}\varepsilon_\kappa + \mu w(z_\kappa, K_\kappa)(1-\varepsilon_\kappa) \\ -\psi_{k'}(s_\kappa; \Psi) \end{array}\right)^{-\nu}$$

$$+p(\psi_{k'}(s_\kappa;\Psi)) + \sum_{z'}\sum_{\varepsilon'}\left[\begin{array}{c} \beta(r(z',K') + (1-\delta))\times \\ \left(\begin{array}{c} (r(z',K') + 1 - \delta)\psi_{k'}(s_\kappa;\Psi) \\ +(1-\tau(z'))w(z',K')\bar{l}\varepsilon' \\ +\mu w(z',K')(1-\varepsilon') \\ -\psi_{k'}(s';\Psi) \end{array}\right)^{-\nu} \times \\ \pi(z',\varepsilon' \mid z_\kappa, \varepsilon_\kappa) \end{array}\right]$$

Individual problem errors (new part):

- Error for $y' = (k')^2$:

$$u_\kappa^* = -\psi_{(k')^2}(s_\kappa; \Psi) + \left(\psi_{k'}(s_\kappa; \Psi)\right)^2$$

Errors only depend only on known things and Ψ

$$
\begin{array}{rcl}
r(z_\kappa, K_\kappa) & = & \alpha z_\kappa (K_\kappa / (\bar{l}(1-u(z_\kappa))))^{\alpha-1} \\
w(z_\kappa, K_\kappa) & = & (1-\alpha) z_\kappa^{\alpha} (K_\kappa / (\bar{l}(1-u(z_\kappa))))^{\alpha} \\
r(z', K') & = & \alpha z' (K' / (\bar{l}(1-u(z'_\kappa)))^{\alpha-1} \\
w(z', K') & = & (1-\alpha) z' (K'/L(z'))^{\alpha} \\
\tau(z) & = & \dfrac{\mu u(z)}{\bar{l}(1-u(z))} \text{ and } \tau(z') = \dfrac{\mu u(z')}{\bar{l}(1-u(z'))} \\
s' & = & \left\{ k', \varepsilon', z', M'_u(1), M'_e(1), M'_u(2), M'_e(2) \right\} \\
 & = & \left\{ \begin{array}{c} \psi_{k'}(s_\kappa; \Psi), \varepsilon', z', \\ M'_u(1), , M'_e(1), M'_u(2), M'_e(2) \end{array} \right\}
\end{array}
$$

Errors only depend only on known things and Ψ

How to get $K', M'_u(1), M'_e(1), M'_u(2), M'_e(2)$ in terms of current state variables?

1. $K' = u(z')M'_u(1) + (1 - u(z'))M'_e(1)$
2. Express $M'_u(j)$ and $M'_e(j)$ in terms of z', z, and $\widehat{M}_u(j)$ and $\widehat{M}_e(j)$
3. Explicitly aggregate to get expressions for $\widehat{M}_u(j)$ and $\widehat{M}_e(j)$

Topics

- non-polynomial basis functions
  - procedure
  - bias correction
- Bond economy

If individual policy rules are not polynomials in levels

- Suppose

$$\begin{aligned} b(k'_u) &= \Psi_{u,0}(s) + \Psi_{u,1}(s)\ b(k_u) \text{ if } \varepsilon = u \text{ and} \\ b(k'_e) &= \Psi_{e,0}(s) + \Psi_{e,1}(s)\ b(k_e) \text{ if } \varepsilon = e, \end{aligned}$$

where $b(k)$ is some function, e.g. $\ln(k)$

If individual policy rules are not polynomials in levels

- Since you need to know K' you need a law of motion for k'.
- Thus, *in addition* to policy rule for $b(k')$ also obtain policy rules for k'.
- How to get policy rule for k'?
  - Use linear approximation to $b(k')$, or
  - Solve two individual policy rules
    - one to solve for aggregate law of motion and
    - one to describe individual behavior

linear approximation for linear spline

- Taking linear approximation when using linear spline is trivial:
    - Get aggregate law of motion for $\widehat{K}_u$ simply by evaluating policy rule of unemployed at $k_u = K_u$
    - Get aggregate law of motion for $\widehat{K}_u$ simply by evaluating policy rule of unemployed at $k_e = K_u$

Bias correction

- If policy rules are *not* polynomials $\Longrightarrow$ inconsistency between
    - individual policy rules and aggregate law of motion
- Estimate of the mean bias can be found from model without aggregate uncertainty

Bias correction

- $\tilde{K}_\varepsilon$ and $\widehat{\tilde{K}}_\varepsilon$ very accurate solution for beginning and end-of-period capital stocks in model without aggregate undertainty
- Xpa aggregate law of motion without bias correction:

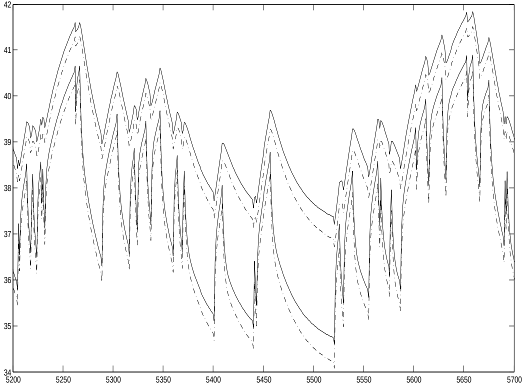
$$\widehat{K}_\varepsilon = \Psi_{\varepsilon,0}(M) + \Psi_{\varepsilon,1}(M)K_\varepsilon.$$

- Bias correction ζ_ε:

$$\zeta_\varepsilon = \widehat{\tilde{K}}_\varepsilon - \Psi_{\varepsilon,0}(\tilde{M}) - \Psi_{\varepsilon,1}(\tilde{M})\tilde{K}_\varepsilon.$$

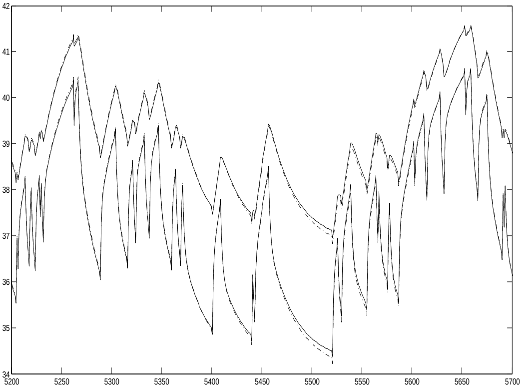
No bias correction

Figure: Simulated values of K_u and K_e without bias correction



With bias correction

Figure: Simulated values of K_u and K_e with bias correction



Would the R2 pick this up?

No, the $R^2 > 0.99997$ in both cases

Imposing equilibrium

- KS economy: at $r_t = z(K_t/L_t)^{\alpha-1}$ equilibrium is ensured for any law of motion for K_t
- For other types of assets this is not that easy
- But *exactly* imposing equilibrium is important
    - errors are unlikely to be exactly average on zero
    - $\Longrightarrow$ errors accumulate and at some point simulation is meaningless

Bond economy

- endowment economy
- borrowing and lending in one and two-period riskless bonds
- penalty functions instead of inequality constraints

Bond economy - equations

$$\frac{q_t^1}{c_{i,t}} = \beta \mathsf{E}_t \frac{1}{c_{i,t+1}} + p(b_{i,t+1}^1)$$

$$\frac{q_t^2}{c_{i,t}} = \beta \mathsf{E}_t \frac{q_{t+1}^1}{c_{i,t+1}} + p(b_{i,t+1}^2)$$

$$c_{i,t} + q_t^1 b_{i,t+1}^1 + q_t^2 b_{i,t+1}^2 = y_{i,t} + b_t^1 + q_t^1 b_t^2,$$

Bond economy - (too) simple approach

- guess law of motion for q_t^1 and q_t^2
- solve individual problem
- problem with simulation: equilibrium is not imposed
- problem with Xpa: equilibrium not imposed off grid points and during simulation

Bond economy - imposing equilibrium

- Instead of solving for $b^j(s_{i,t}, m_t)$ solve for $b^j(q_t^j, s_{i,t}, m_t)$
- Solve for q_t^j from

$$\int b^j(q_t^j, s_{i,t}, m_t)di = 0$$

- How do I get these $b^j(q_t^j, s_{i,t}, m_t)$?

Bond economy - imposing equilibrium

- Use model equations:

$$\begin{aligned}
\frac{q_t^1}{c_{i,t}} &= \beta \mathsf{E}_t \frac{1}{c_{i,t+1}} + p(b_{i,t+1}^1) \\
\frac{q_t^2}{c_{i,t}} &= \beta \mathsf{E}_t \frac{q_{t+1}^1}{c_{i,t+1}} + p(b_{i,t+1}^2) \\
c_{i,t} + q_t^1 b_{i,t+1}^1 + q_t^2 b_{i,t+1}^2 &= y_{i,t} + b_t^1 + q_t^1 b_t^2
\end{aligned}$$

- and add the following two equations that define d_{t+1}^1 and d_{t+1}^2

$$b_{i,t+1}^1 + q_t^1 = d_{t+1}^1 \text{ and}$$
$$b_{i,t+1}^2 + q_t^2 = d_{t+1}^2$$

Bond economy - imposing equilibrium

- This gives the following solutions
  - $q(m_t)$
  - $b^j(s_{i,t}, m_t)$
  - $d^j(s_{i,t}, m_t)$
- Take **none** of these literally. Except use
  - $b^j_{t+1} = b^j(q^j_t, s_{i,t}, m_t) = d^j(s_{i,t}, m_t) - q^j_t$

Bond economy - imposing equilibrium

- Imposing equilibrium

$$\int b^j(q_t, s_{i,t}, m_t)di = 0$$

gives

$$q_t^j = \int d^j(s_{i,t}, m_t)di$$

Bond economy - imposing equilibrium

- Alternative definitions for $d^j(\cdot)$ are possible
- !!!! But one does need that

$$\frac{\partial b^j(q_t, s_{i,t}, m_t)}{\partial q_t} < 0$$

that is, you have a demand equation.

Something practical

- The following slides work out a particular case
- The individual policy rule is of a simpler form than used in discussing the general case above
- Given this simpler form the slides go through the steps of XPA

Something practical

- Suppose that

$$
\begin{aligned}
z_t &= \bar{z} + \rho_z z_{t-1} + e_{z,t} \quad \text{with } \mathsf{E}\left[e_{z,t}\right] = \sigma_z^2 \\
\varepsilon_{i,t} &= \bar{\varepsilon}(1-\rho_\varepsilon) + \rho_\varepsilon \varepsilon_{t-1} + e_{\varepsilon,t} \quad \text{with } \mathsf{E}\left[e_{\varepsilon,t}\right] = \sigma_\varepsilon^2
\end{aligned}
$$

- Note that

$$
\mathsf{E}\left[\varepsilon_{i,t}\right] = \bar{\varepsilon}
$$

Something practical

- Individual policy rule is higher-order

$$k' = \Psi_0^* + \Psi_k k + \Psi_\varepsilon \varepsilon + \Psi_{k\varepsilon} k\varepsilon + \Psi_{u,z} z + \Psi_{u,K} K$$

- This can be written as

$$k' = \Psi_0 + \Psi_k k + \Psi_\varepsilon \varepsilon + \Psi_{k\varepsilon} \rho k \varepsilon_{-1} + \Psi_{k\varepsilon} e_\varepsilon + \Psi_{u,z} z + \Psi_{u,K} K$$

Something practical

Aggregation of

$$k' = \Psi_0 + \Psi_k k + \Psi_\varepsilon \varepsilon + \Psi_{k\varepsilon} \rho k \varepsilon_{-1} + \Psi_{k\varepsilon} e_\varepsilon + \Psi_z z + \Psi_K K$$

gives

$$K' = (\Psi_0 + \Psi_\varepsilon \bar{\varepsilon}) + (\Psi_k + \Psi_{u,K}) K + \Psi_{k\varepsilon} \rho M_{k\varepsilon} + \Psi_{u,z} z$$

where

$$M_{k\varepsilon} = \int k \varepsilon_{-1} dF(k, \varepsilon_{-1})$$

Something practical

- Aggregate state variables: $z, K, M_{k\varepsilon_{-1}}$

- Why not K_u and K_e separately?
- Why don't we have to use transition laws?
- How we get law of motion for $M'_{k\varepsilon}$?
- **Answer**: define

$$y' = k'\varepsilon \approx \widetilde{\Psi}_0 + \widetilde{\Psi}_k k + \widetilde{\Psi}_\varepsilon \varepsilon + \widetilde{\Psi}_{k\varepsilon} \rho k \varepsilon_{-1} + \widetilde{\Psi}_{k\varepsilon} e_\varepsilon + \widetilde{\Psi}_z z + \widetilde{\Psi}_K K$$

Something practical

Aggregation of

$$y' = \widetilde{\Psi}_0 + \widetilde{\Psi}_k k + \widetilde{\Psi}_\varepsilon \varepsilon + \widetilde{\Psi}_{k\varepsilon} \rho k \varepsilon_{-1} + \widetilde{\Psi}_{k\varepsilon} e_\varepsilon + \widetilde{\Psi}_z z + \widetilde{\Psi}_K K$$

gives

$$M'_{k\varepsilon} = \left(\widetilde{\Psi}_0 + \widetilde{\Psi}_\varepsilon \bar{\varepsilon} \right) + \left(\widetilde{\Psi}_k + \widetilde{\Psi}_{u,K} \right) K + \widetilde{\Psi}_{k\varepsilon} \rho M_{k\varepsilon} + \widetilde{\Psi}_{u,z} z$$

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