Solving Models with Heterogeneous Agents (not with KS or Xpa)

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Algorithms other than KS & XPA

- Den Haan (1996)
  - very simple
- Reiter (2008)
  - smart hybrid; projection to deal with idiosyncratic risk, perturbation to keep cost low
- Roca & Preston (2007)
  - pure perturbation, thus fast
- Algan, Allais, & Den Haan (2008)
  - pure projection; can handle transition from non-typical cross-sectional distributions

Den Haan 1996

- Cross-sectional distribution characterized with finite set of moments
- No *explicit* approximate law of motion for aggregate variables
  - $\Longrightarrow$ no additional inaccuracies introduced
- Full simulation method
  - $\Longrightarrow$ easy
  - $\Longrightarrow$ computationally expensive

$$c_{i,t}^{1-\nu} = p_N(s_{i,t};\psi)$$

solve $k_{i,t+1}$ from budget constraint

$$\begin{array}{ll} \text{if } k_{i,t+1} \geq 0 & \text{done} \\ \text{if } k_{i,t+1} < 0 & \left\{ \begin{array}{c} k_{i,t+1} = 0 \\ \text{solve } c_{i,t} \text{ from bc} \end{array} \right. \end{array}$$

- Conditional expectation of *individual* $\approx p_N(s_{i,t};\psi)$
- $s_{i,t} = \{k_{i,t}, e_{i,t}, z_t,\text{info about cross-sectional distribution}\}$

Simulating a panel for given value psi

- generate aggregate productivity $\{z_t\}_{t=1}^{T}$
- start in $t = 1$ with cross-section of I agents
  - Thus, $k_{i,t}$ and $e_{i,t}$ known at $t = 1$
- use cross-section to calculate K_t and other moments
- use K_t and z_t to calculate r_t and w_t
- for each agent calculate $k_{i,t+1}$
- for each agent draw new $e_{i,t}$ and go to the next period

Simulating a panel for given value psi

- To update individual problem:
  - you only need to variables for 1 agent
- But individual choices depend on aggregates
  - $\Longrightarrow$ you need a panel

Updating individual law of motion

If $k_{t+1} > 0$ $\begin{cases} c_t^{-\nu} = \mathsf{E}_t \left[\beta c_{t+1}^{-\nu}(r_{t+1} + 1 - \delta)\right] \\ \text{solve } k_{t+1} \text{ from budget constraint} \end{cases}$

- collect observations with $k_{t+1} > 0$
- regress $\beta c_{t+1}^{-\nu}(r_{t+1} + 1 - \delta)$ on $p_n(s_t;\psi) \Longrightarrow \widehat{\psi}$
- aggregate law of motion taken care of (implicitly)
- update ψ using weighted average of $\widehat{\psi}$ and old ψ

Updating aggregate law of motion

- Not needed
- In the simulation, aggregate variables are constructed by explicitly aggregating the values across I individuals
- Thus, no approximation needed to describe law of motion of aggregate variables

Advantages & Disadvantages

- even simpler than KS
- even construction of individual policy rules done using simulation methods
    - disadvantages of simulation methods can be lessened/avoided using the improvements suggested by Maliar, Maliar, & Judd

Imposing equilibrium in panel

- It is important to impose equilibrium
  - unless the average deviation (across periods) is exactly zero the deviation will accumulate and increase without bound
- True for any simulation procedure (including KS)
- Equilbrium automatically imposed in capital economy

Tricks to impose equilibrium

Suppose, you want to impose

$$\sum_{i=1}^{I} b(s_{i,t}) = 0$$

Then, get approximation for $d(s_{i,t})$, where

$$d(s_{i,t}) = q_t - b(s_{i,t})$$

and get q_t from

$$q_t = \sum_{i=1}^{I} d(s_{i,t})/I_t$$

and $b_{i,t}$ from

$$b_{i,t} = q_t - d(s_{i,t})$$

Tricks to impose equilibrium

You may think that

$$d(s_{i,t}) = q_t - b(s_{i,t})$$

and

$$d(s_{i,t}) = q_t + b(s_{i,t})$$

both work.

But stability properties of the algorithm can be very different

Tricks to impose equilibrium

$$\text{If } \frac{\sum_{i=1}^{I} b_i}{I} = \bar{b} > 0,$$

then the bond price is too low.

$$\text{If } d(s_{i,t}) = q_t - b(s_{i,t})$$

then

$$q^{\text{new}} = \frac{\sum_{i=1}^{I} d(s_{i,t})}{I} = \frac{\sum_{i=1}^{I} q^{\text{old}}}{I} - \bar{b}$$

and

$$q^{\text{new}} > q^{\text{old}}$$

as needed

Crucial insights of Reiter (2008)

- idiosyncratic uncertainty large
  - $\Longrightarrow$ individual problem likely to be non-linear
  - $\Longrightarrow$ perturbation probably bad idea
- with idiosyncratic and without aggregate uncertainty
  - still doable even when problem is highly non-linear
- aggregate uncertainty small
  - $\Longrightarrow$ aggregate problem probably easy
  - $\Longrightarrow$ perturbation likely to work

Crucial insights of Reiter (2008)

- Combine perturbation and projection

Perturbation combined with Projection

- With idiosyncratic risk you can be quite far from steady state ($\sigma_e = \sigma_z = 0$)
- Idea of Reiter: Focus on steady state implied by $\sigma_z = 0$ and $\sigma_e > 0$
- $\sigma_a = 0 \Longrightarrow$ cross-sectional distribution doesn't change over time $\Longrightarrow$ pretty standard problem to solve

Elements

1. A *numerical* solution to the model:

$$k_{i,t+1} = P_N(e_{i,t}, k_{i,t}, z_t, m_t; \boldsymbol{\lambda_k}),$$

   m_t is a characterization of the distribution

2. λ_k **should pin down everything**.
   In particular, $P_N(\cdot; \lambda_k)$ pins down aggregate law of motion

$$m_{t+1} = \Gamma_{\lambda_k}(z_{t+1}, z_t, m_t)$$

   This requires that m_t describes complete distribution:

   - m_t can be CDF values at a fine grid
   - m_t is set of moments *and* distributional assumption as in AAD is made

Rewrite the policy function

- Rewrite the *numerical* solution to the model as

$$k_{i,t+1} = P_n(e_{i,t}, k_{i,t}; \boldsymbol{\lambda_{k,t}})$$

with

$$\boldsymbol{\lambda}_{k,t} = \lambda_k(z_t, m_t)$$

Notation & grid

- $\tilde{s} = [z, m]$
- ε_j and κ_j: employment status and capital at grid point j
- Dimension of $\boldsymbol{\lambda}_{k,\boldsymbol{t}} = n^{\#}_{\lambda_k}$
  - $P_N(\cdot)$ is 2$^{\text{nd}}$-order complete & $\tilde{s}$ is $3 \times 1 \Longrightarrow n^{\#}_{\lambda_k} = 6$
  - number of grid points $= n^{\#}_{\text{grid}} \geq n^{\#}_{\lambda_k}$
- no grid for $\tilde{s}$!!!

Model equation at grid points

Setting $\delta = 1$ for simplicity

$$\frac{1}{\left(r(\tilde{s})\right)\kappa_j + w(\tilde{s})\varepsilon_j \bar{l} - P_N(\varepsilon_j, \kappa_j; \lambda_k(\tilde{s}))} =$$

$$\mathsf{E}\frac{\beta\left(r(\tilde{s}')\right)}{\left[\begin{array}{c} \left(r(\tilde{s}')\right)P_N(\varepsilon_j, \kappa_j; \lambda_k(\tilde{s})) + w(\hat{s})\varepsilon_{+1}\bar{l} \\ -P_N(\varepsilon_{+1}, P_N(\varepsilon_j, \kappa_j; \lambda_k(\tilde{s})); \lambda_k(\tilde{s}_{+1})) \end{array}\right]}$$

Three more useful equations

1. $r(\tilde{s}) = \alpha z \left(K/\bar{l}\right)^{\alpha-1}$
2. $w(\tilde{s}) = (1-\alpha) z \left(K/\bar{l}\right)^{\alpha}$
3. $m_{+1} = \Gamma_{\lambda_k}(z_{+1}, z, m)$
    - m is histogram in Reiter
    - $\Longrightarrow \Gamma_{\lambda_k}$ is fully known (see slides on simulation with continuum of agents)

Mental break

- Have I really done anything?
- Not much
    - constructed a grid
    - construct a system with individual choices substituted out

Mental break

- Suppose $n^{\#}_{\lambda_k} = 9$ and there are 9 grid points
- Then I have the following type of system

$$\underset{9\times 1}{F(\lambda_k(\tilde{s}), \tilde{s})} = \underset{9\times 1}{0}$$

system above & possible weighting (if $n^{\#}_{\text{grid}} \geq n^{\#}_{\lambda_k}$) $\Longrightarrow F(\cdot)$. Thus,

  - $F(\cdot)$ known
  - $\lambda_k(\tilde{s})$ unknown

- Standard perturbation system

Perturbation system

- What is fixed and what are the state variables?

Perturbation system

- What is fixed and what are the state variables?
  - state variables: $\tilde{s}$

Perturbation system

- What is fixed and what are the state variables?
    - state variables: $\tilde{s}$
- What are not the state variables?

Perturbation system

- What is fixed and what are the state variables?
  - state variables: $\tilde{s}$
- What are not the state variables?
  - not state variables: ε and κ

Standard perturbation problem

- Let σ_z characterize the uncertainty in z
- Let $h_{\lambda_k}(z, m; \sigma_z)$ be the Taylor expansion of $\lambda_k(\tilde{s})$

$$\begin{array}{c} h_{\lambda_k}(z, m; \sigma_z) \\ = \begin{array}{c} h_{\lambda_k}(\bar{z}, \bar{m}; 0) + h_{\lambda_k, z}(z - \bar{z}) + h_{\lambda_k, m}(m - \bar{m}) + h_{\lambda_k, \sigma_z}\sigma_z \\ + \; h_{\lambda_k, z^2}(z - \bar{z})^2/2 + h_{\lambda_k, m^2}(m - \overline{m})^2/2 + h_{\lambda_k, \sigma_z^2}\sigma_z^2/2 \\ + \text{ second-order cross products} \\ + \; \cdots \end{array} . \end{array}$$

Standard perturbation problem

Solve for coefficients $h_{\lambda_k,z}$, $h_{\lambda_k,m}$, h_{λ_k,σ_z}, h_{λ_k,z^2}, etc. by sequentially differentiating

$$F(\lambda_k(\tilde{s}), \tilde{s}) = 0$$

Dimensions in Reiter (2008)

- Reiter uses fine histogram to characterize CDF
  - $\Longrightarrow$ dimension of m typically high
  - $> 1,000$ in Reiter (2008)
  - $\Longrightarrow$ higher-order perturbation becomes tough

Alternative to histogram

Suppose

1. m_t consists of N_m moments **and**
2. functional form of cross-sectional density of e_i and k_i is known

$\Longrightarrow$

$m_{+1} = \Gamma_{\lambda_k}(z_{+1}, z, m)$ is known as well

Alternative to histogram

Suppose

1. m consists of 2 moments **and**
2. functional form of cross-sectional density of e_i and k_i is Normal

- Distribution is endogenous so Normal could be bad choice
- Choice of functional form for cross-sectional density less important when m includes more moments

AAD - Key step

- $$\text{moments} \iff \text{functional form density}$$
- $$\text{trivial for Normal}$$
- $$\text{can this be generalized?}$$

AAD - Flexible cross-sectional density

$$P_{N_m}(k,\rho) = \rho_0 \exp\left(\begin{array}{c} \rho_1 \left[k - m(1)\right] + \\ \rho_2 \left[(k - m(1))^2 - m(2) \right] + \cdots + \\ \rho_{N_m} \left[(k - m(1))^{N_m} - m(N_m) \right] \end{array} \right).$$

where $m(n)$ is the n^{th} uncentered moment

- Goal: find the ρs such that values of moments match implied moments
- This particular functional form $\Longrightarrow$

$$\min_{\rho_1,\rho_2,\cdots,\rho_{Nm}} \int_0^\infty P_{N_m}(k,\rho)dk.$$

and this is a convex problem

KS-type economy

- i.i.d. idiosyncratic productivity

$$
\begin{aligned}
e_{i,t} &= 1-\Delta_e \text{ with probability} = 0.5 \\
e_{i,t} &= 1+\Delta_e \text{ with probability} = 0.5
\end{aligned}
$$

- aggregate shock

$$
\ln(z_t) = \rho_z \ln(\ln z_{t-1}) + u_{z,t}
$$

- complete depreciation to simplify equations

Key decisions

- Complete cross-sectional distribution is pinned down by the average values of

$$k_{i,t\_1} \text{ and } k_{i,t-1}^2$$

aggregate across *all* agents.

- For the *complete* distribution to be pinned down by only two moments we need a strong functional form assumption. Just to explain the method assume it is Normal
  - If we have more moments then the functional form assumption matters less

Key decisions

- Individual policy rule:

$$
\begin{aligned}
k_i &= P(k_{i,-1}, e, \tilde{s}, \lambda) \\
&= P(k_{i,-1}, e, \lambda(\tilde{s})) \\
k_i &= \lambda_0(\tilde{s}) + \lambda_1(\tilde{s}) k_{i,-1} + \lambda_2(\tilde{s}) e_i \\
&\quad + \lambda_3(\tilde{s}) k_{i,-1}^3 + \lambda_4(\tilde{s}) k_{i,-1}^2 e_i + \lambda_5(\tilde{s}) k_{i,-1}^3
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{s} &= \left[z, \int k_{i,-1} di, \int k_{i,-1}^2 di \right] \\
&= [z, K_{-1}, M_{-1}]
\end{aligned}
$$

Goal

- Find approximation to individual policy rule
- That is, find approximation to the five functions $\lambda_j(\tilde{s})$
- To use perturbation we need 6 functional equations, equations that hold for all $\tilde{s}$

$$F(\lambda_0(\tilde{s}), \lambda_1(\tilde{s}), \lambda_2(\tilde{s}), \lambda_3(\tilde{s}), \lambda_4(\tilde{s}), \lambda_5(\tilde{s}), \tilde{s}) = 0$$

Constructing F(.) - grid

- Construct a grid of e_i and $k_{i,-1}$ with six grid points
  - $\varepsilon_1, \varepsilon_2$ for e_i and $\kappa_1, \kappa_2, \kappa_3$ for $k_{i,-1}$

Constructing F(.) - Euler eq

$$\frac{1}{r(\tilde{s})\kappa_j + w(\tilde{s})\varepsilon_j - P(\varepsilon_j, \kappa_j, \lambda(\tilde{s}))}$$

$$= \mathsf{E}\sum_{l_e=1}^{2} \frac{\beta r(\tilde{s}_{+1})}{\begin{bmatrix} r(\tilde{s}_{+1})P(\varepsilon_j, \kappa_j, \lambda(\tilde{s}_{+1})\kappa_j + w(\tilde{s})\varepsilon_{l_e} \\ -P(\varepsilon_{l_e}, -P(\varepsilon_j, \kappa_j, \lambda(\tilde{s}), \lambda(\tilde{s}_{+1})) \end{bmatrix}}$$

Helpful equations

$$
\begin{aligned}
r(\tilde{s}) &= \alpha z K^{\alpha-1} L^{1-\alpha} \\
w(\tilde{s}) &= (1-\alpha) z K^{\alpha} L^{\alpha} \\
L &= 1
\end{aligned}
$$

Do I have six equations?

- How to deal with $\tilde{s}_{+1}$?
  - $\ln(z_{t+1}) = \rho_z \ln(\ln z_t) + u_{z,t+1}$
  - $K = \int k_i di = \int P(e_i, k_i, \lambda(\tilde{s}) f(k_i; K_{-1}, M_{-1}) dk_i$
  - $M = \int k_i^2 di = \int (P(e_i, k_i, \lambda(\tilde{s}))^2 f(k_i; K_{-1}, M_{-1}) dk_i$
  - $f(k_i; K_{-1}, M_{-1})$ is Normal so use Gaussian quadrature

Do I have six equations?

- How to deal with E over u_{t+1}?
- Again use Gaussian quadrature

Conclusion:

- I have six very messy equations that hold for any value of $\tilde{s}$
- You could even give this to Dynare :)

Pure perturbation (Roca & Preston)

- KS model has to be modified a little
  - discrete support is likely to be difficult $\Longrightarrow$ continuous support
  - borrowing constraint is definitely difficult $\Longrightarrow$ penalty function
- Perturbation around solution when there is no aggregate and no idiosyncratic uncertainty
- Capital is in levels (not in logs or any other transformation)

Model modifications

- $e_{i,t}$ can take on continuum of values

$$\begin{aligned} e_{i,t+1} &= (1-\rho_e) + \rho_e e_{i,t} + \varepsilon^e_{i,t+1} \\ \varepsilon^e_{i,t+1} &\sim N(0,\sigma^2) \\ \mathsf{E} e_{i,t} &= 1 \Longrightarrow L = 1 \end{aligned}$$

- Continuous penalty term when capital is getting smaller. FOCs:

$$\begin{aligned} c_{i,t}^{-\nu} &= \mathsf{E}_t \left[-2\phi k_{i,t+1}^{-3} + c_{i,t+1}^{-\nu} (r^k_{t+1} + 1 - \delta) \right] \\ k_{i,t+1} &= (1-\delta) k_{i,t} + r_t k_{i,t} + w_t e_{i,t} \bar{l} - c_{i,t} \end{aligned}$$

State variables and order of perturbation

- Individual state variables $s_{i,t} = \{k_{i,t}, e_{i,t}, \tilde{s}_t\}$
- Again a limited set of moments is used as state variables
- As with Xpa, the elements of $\tilde{s}_t$ depend on approximation order
  - First order: $\tilde{s}_t = \{a_t, K_t\}$
  - Second order: $\tilde{s}_t = \{a_t, K_t, \Phi_t, \Psi_t\}$
  -

$$\begin{aligned}
K_t &= \int_0^1 (k_{i,t} - K_t)\, di, \\
\Phi_t &= \int_0^1 (k_{i,t} - K_t)^2\, di, \text{ and} \\
\Psi_t &= \int_0^1 (k_{i,t} - K_t)\,(e_{i,t} - \mu_e) di.
\end{aligned}$$

What to solve for?

Solve for $h_v(s_{i,t}, \sigma)$ with $v \in \{c, k, K, \Phi, \Psi\}$

What to solve for?

$$\frac{1}{h_c\left(s_{i,t},\sigma\right)} = \beta E_t\left[-2\phi h_k(s_{i,t},\sigma)^{-3} + \frac{(r_{t+1}+1-\delta)}{h_c(s_{i,t+1})}\right]$$

$$h_k(s_{i,t},\sigma) = (1-\delta)k_{i,t} + r_t k_{i,t} + w_t e_{i,t}\bar{l} - h_c(s_{i,t},\sigma)$$

$$K_{t+1} = h_K(\tilde{s}_t,\sigma) = \int_0^1 h_k(s_{i,t},\sigma)di$$

$$\Phi_{t+1} = h_\Phi(\tilde{s}_t,\sigma) = \int_0^1 \left(h_k(s_{i,t},\sigma) - \bar{k}\right)^2 di$$

$$\Psi_{t+1} = h_\Psi(\tilde{s}_t,\sigma) = \int_0^1 \left(h_k(s_{i,t},\sigma) - \bar{k}\right)\left(e_{i,t} - \mu_e\right) di$$

$$r_t = \alpha a_t \left(K_t/\bar{l}\right)^{\alpha-1}$$
$$w_t = (1-\alpha) a_t \left(K_t/\bar{l}\right)^{\alpha}$$

$$s_{i,t+1} = \left\{ \begin{array}{c} h_k(s_{i,t}), 1-\rho_e+\rho_e e_{i,t}+\varepsilon_{i,t+1}, 1-\rho+\rho a_t+\varepsilon_{a,t+1}, \\ h_K(\tilde{s}_t,\sigma), h_\Phi(\tilde{s}_t,\sigma), h_\Psi(\tilde{s}_t,\sigma) \end{array} \right\}$$

- system expressed in period t variables and period $t+1$ shocks
- we now have a perturbation system
  - differentiating system gives values derivatives of h_v
  - from these we get coefficients of Taylor expansions

Comments

- There is no approximation in system specified so far (which is good)
- w_t and r_t depend on mean of the *level* of capital
- We have an exact equation for the mean *because capital is in levels* (and not in logs)
- Capital in logs $\Longrightarrow$ one has to approximate aggregation definition
  - e.g., linearizing around steady state
  - given dispersion in capital levels this may not be accurate

Moments and order of perturbation (first-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t, and K_{t+1}
- First-order perturbation $\Longrightarrow$ linear policy rules $\Longrightarrow$ K_{t+1} only depends on K_t, a_t,and nothing else (mean of $e_{i,t}$ is constant through time)

Moments and order of perturbation (second-order)

- Agents obviously care about $k_{i,t}$, $e_{i,t}$, a_t, and K_{t+1}
- Second-order perturbation $\Longrightarrow$ agent's policy rules depend on $\left(k_{i,t}-\bar{k}\right)^2$ and $\left(k_{i,t}-\bar{k}\right)\left(e_{i,t}-\mu_e\right)$
  - $\Longrightarrow K_{t+1}$ depends on K_t, Φ_t, Ψ_t, and z_t,and nothing else
- Does second-order perturbation include terms like
  - $\left(k_{i,t}-\bar{k}\right)^2 K_t$?
  - $K_t \Psi_t$?
  - $\Psi_t \Phi_t$?

Dealing with transitions

- KS, Xpa, Den Haan 1996, Roca Preston, & Reiter (hybrid) focus on
  - small changes aggregate variables close to steady state
  - behavior aggregate variables in a *typical* simulation
- This means they cannot deal with
  - transition after a *one-time* and *unforeseen* redistribution of capital
  - destruction of capital

Simple analogy

- Projection solution of the neoclassical growth model gives

$$k' \approx p_N(k, a; \psi)$$

- By using a wide enough grid for k and a and a rich enough approximating function one ensures accuracy for *all* values of k and a inside the grid including those not encountered in a simulation
- In solving heterogeneous agent models can you attain accuracy for any cross-sectional distribution?

Algan, Allais, and Den Haan (2008)

- use projection methods and quadrature techniques as much as possible
  - $\Longrightarrow$ construct a grid for the aggregate state variables including moments
- calculate next-period's moments using quadrature techniques
  - quadrature integration requires functional form for the distribution
  - use flexible functional form and link moments with polynomial's coefficients

AAD - Key step

$$\text{moments} \iff \text{functional form density}$$

AAD - Flexible cross-sectional density

$$P_N(k,\rho) = \rho_0 \exp\left(\begin{array}{c} \rho_1 \left[k - m(1)\right] + \\ \rho_2 \left[(k - m(1))^2 - m(2)\right] + \cdots + \\ \rho_N \left[(k - m(1))^N - m(N)\right] \end{array}\right).$$

where $m(n)$ is the n^{th} uncentered moment

- Goal: find the ρs such that values of moments match implied moments
- This particular functional form $\Longrightarrow$

$$\min_{\rho_1,\rho_2,\cdots,\rho_N} \int_0^\infty P_N(k,\rho)dk.$$

and this is a convex problem

AAD #1: specify aggregate law of motion

- construct grid for aggregate state variables
- calculate next period's moments
- do projection step to calculate aggregate law of motion
- construct grid for individual agent (includes aggregate state variables)
- solve individual problem (for given aggregate law of motion)
- iterate between aggregate and individual problem

AAD #2: do not specify aggregate law of motion

- construct grid for individual problem (includes aggregate state variables)
- calculate next period's mean (and thus r_t and w_t) directly using quadrature techniques
- Solve individual problem

Problem with algorithm so far

- Moments fulfill two roles
  - state variable
  - get the shape of the distribution right
- To get shape right, you need several moments
  - $\Longrightarrow$ you need several state variables
- Solution:
  - use limited set of moments as state variables
  - use additional higher-order moments as reference moments for good shape

Getting the distribution right

- Suppose you only use the mean capital stock as a state variable
- But use $N(K, \sigma_K^2)$ as the cross-sectional distribution
- You still have to find σ_K^2
- AAD get reference moments from a simulation
- reference moments could depend on the state, i.e., $N(K, (\sigma_K(\tilde{s}_t))^2$

References

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  - stresses importance of using $\mathsf{E}_t\left[y_{t+1}\right]$ insteadof y_{t+1} in PEA
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  - give seveal useful suggestions to make PEA better