Value Function Iteration versus Euler equation methods

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Overview

1. How to do value function iteration (VFI)
2. VFI versus Euler equation methods
   1. convergence
   2. speed
   3. complex problems

Bellman equation

$$V(x) = \max_{x_{+1} \in \Gamma(x)} U(x, x_{+1}) + \mathsf{E}_t \left[\beta V(x_{+1}) \right]$$

Essence of VFI

- $V^i(x)$: flexible functional form
  - piecewise linear (or higher-order spline)
  - discrete valued function (if $\Gamma(x)$ has $\chi < \infty$ elements)
  - quadratic (or higher-order polynomial)

- $V^{i+1}(x)$ is obtained from

$$V^{i+1}(x) = \max_{x_{+1} \in \Gamma(x)} U(x, x_{+1}) + E_t \left[\beta V^i(x_{+1}) \right]$$

Essence of VFI

- This works in general
- However, on a computer the functional form of $V^i(x)$ must stay the same
  (so computer can store coefficients characterizing function)

Possible ways to implement VFI

1. Linear-Quadratic
   - $U(\cdot)$ is quadratic and constraints are linear
     $\Longrightarrow V^i(\cdot)$ would remain quadratic
   - !!! To get a true first-order approximation to policy function you cannot take linear approximation of constraints
     $\Longrightarrow$ either get rid of constraint by substitution or use the "correct" LQ approximation (see perturbation slides)

2. Discrete grid $\Longrightarrow \Gamma(x)$ and $V(x)$ have finite # of elements

Possible ways to implement VFI

3. Piecewise linear

- *choices* are no longer constrained to be on grid
- $V^i(\cdot)$ is characterized by function values on grid
- Simply do maximization on grid

4. Regular polynomial

- *choices* are no longer constrained to be on grid
- calculate values V on grid
- obtain V^{i+1} by fitting polynomial through calculated point

Convergence

- There are several convergence results for VFI
- Some such results for Euler equation methods
  - but you have to do it right (e.g. use time & not fixed-point iteration)
- But especially for more complex problems, VFI is more likely to converge

Speed; algorithm choice

- VFI: because of the max operator you typically can only iterate
  - slow if discount factor is close to 1
- Euler equation method have more options
  - calculating fixed point directly with equation solver typically faster

Speed; impact choices on V & Euler

VFI tends to be slow in many typical economic applications

- Reason: value function is flat $\Longrightarrow$ hard to find max
  - important to be aware of this
  - Krusell and Smith (1996) show that utility loss of keeping capital stock constant is minor in neoclassical growth model
  - But shouldn't a flat utility function be problematic for Euler eq. methods as well?

Speed; impact choices on V & Euler

Example to show Euler eq. methods less affected by flatness

$$\max_{x_1,x_2} x_1^{1-\nu} + x_2^{1-\nu}$$
$$\text{s.t. } x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Speed; impact choices on V & Euler

Consider a *huge* move away from optimum

ν	$u(1,1)$	$u(2,0)$	consumption equivalent loss
0.01	2	1.9862	0.7%
0.001	2	1.9986	0.07%

Speed; impact choices on V & Euler

First-order condition:

$$\left(\frac{x_1}{x_2}\right)^{-\nu} = 1 \text{ or } x_1 = 1^{-1/\nu} \times x_2$$

Marginal rates of substitution:

ν	$x_1 = x_2 = 1$	$x_1 = 2, x_2 = 0$
0.01	1	∞
0.001	1	∞

Dealing with complex problems

- Both VFI and Euler-equation methods can deal with inequality constraints
- Euler equations require first-order conditions to be sufficient
  - this requires concavity (utility function) and convex opportunity set
  - this is not always satisfied

Non-convex problem - example

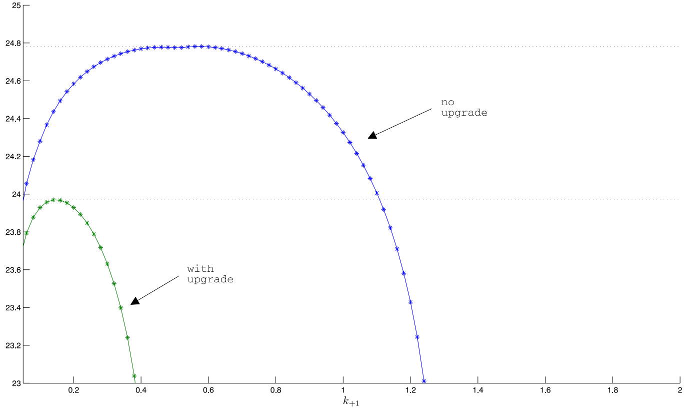
Environment:

- Two technologies:
  - $y_t = k_t^\alpha$
  - $y_t = Ak_t^\alpha$ with $A > 1$
- Higher-productivity technology can be used after paying a one-time cost ψ

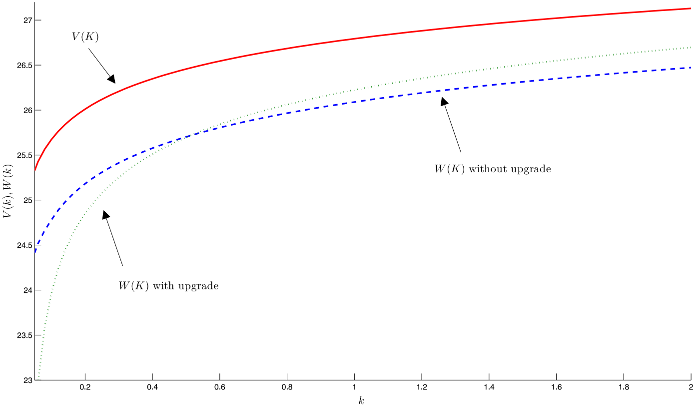
Non-convex problem - example

$$
\begin{aligned}
W(k) &= \max \left\{ \begin{array}{c} \max_{k_{+1}} k^{\alpha} - k_{+1} + \beta W\left(k_{+1}\right), \\ \max_{k_{+1}} k^{\alpha} - k_{+1} - \psi + \beta V\left(k_{+1}\right) \end{array} \right\} \\
V(k) &= \max_{k_{+1}} A k^{\alpha} - k_{+1} + \beta V\left(k_{+1}\right)
\end{aligned}
$$

RHS Bellman equation for low capital stock (k=0.1)



Ultimate value function



References

- Slides on perturbation; available online.
- Slides on projection methods; available online.
- Judd, K. L., 1998, Numerical Methods in Economics.
- Krusell, P. & A. Smith, 1996. Rules of thumb in macroeconomic equilibrium A quantitative analysis, Journal of Economic Dynamics and Control.
- Rendahl, P., 2006, Inequality constraints in recursive economies.
  - shows that time-iteration converges even in the presence of inequality constraints