Parameterized Expectations Algorithm

Wouter J. Den Haan
London School of Economics

Overview

- Two PEA algorithms
- Explaining simulations PEA
- Advantages and disadvantages
- Improvements of Maliar, Maliar & Judd
- PEA to introduce learning

Model

$$
\begin{aligned}
c_t^{-\nu} &= \mathsf{E}_t\left[\beta c_{t+1}^{-\nu}\left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta\right)\right] \\
c_t + k_{t+1} &= z_t k_t^{\alpha} + (1-\delta)\, k_t \\
\ln(z_{t+1}) &= \rho \ln(z_t) + \varepsilon_{t+1} \\
&\quad \varepsilon_{t+1} \sim N(0, \sigma^2) \\
&\quad k_1, z_1 \text{ given}
\end{aligned}
$$

k_t is beginning-of-period t capital stock

Two types of PEA

1. As a standard projections algorithm:
   1. parameterize $\mathsf{E}_t[\cdot]$ with $P_n(k_t, z_t; \eta_n)$
      (note that $\mathsf{E}_t[\cdot]$ is a function of the usual state variables.)
   2. solve c_t from
      $$c_t = (P_n(k_t, z_t; \eta_n))^{-1/\nu}$$
      and k_{t+1} from budget constraint
   3. $\Longrightarrow$ only difference is that $\mathsf{E}_t[\cdot]$ is parameterized instead of consumption or capital choice.
2. Simulation PEA (stochastic and non-stochastic)

Stochastic PEA based on simulations

1. Simulate $\{z_t\}_{t=1}^{T}$
2. Let η_n^1 be initial guess for η_n

Stochastic PEA

3. Iterate until η_n^i converges using following scheme
   1. Generate $\{c_t, k_{t+1}\}_{t=1}^{T}$ using

$$\begin{aligned} c_t^{-\nu} &= P_n(k_t, z_t; \eta_n^i) \\ k_{t+1} &= z_t k_t^{\alpha} + (1-\delta) k_t - c_t \end{aligned}$$

   2. Generate $\{y_{t+1}\}_{t=1}^{T-1}$ using

$$y_{t+1} = \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right)$$

   3. Let

$$\hat{\eta}_n^i = \arg\min_{\eta} \sum_{t=T_{\text{begin}}}^{T} \frac{\left(y_{t+1} - P_n(k_t, z_t; \eta)\right)^2}{T}$$

   4. Update using

$$\eta_n^{i+1} = \omega \hat{\eta}_n^i + (1-\omega) \eta_n^i \text{ with } 0 < \omega \leq 1$$

Stochastic PEA

- $T_{\text{begin}} >> 1$ (say 500 or 1,000)
  - ensures possible bad period 1 values don't matter
- $\omega < 1$ improves stability
  - ω is called "dampening" parameter

Stochastic PEA

- Idea of regression:

$$y_{t+1} \approx P_n(k_t, z_t; \eta) + u_{t+1},$$

- u_{t+1} is a prediction error $\Longrightarrow$ u_{t+1} is orthogonal to regressors
- Suppose

$$P_n(k_t, z_t; \eta) = \exp\left(a_0 + a_1 \ln k_t + a_2 \ln z_t\right).$$

- You are *not* allowed to run the linear regression

$$\ln y_{t+1} = a_0 + a_1 \ln k_t + a_2 \ln z_t + u^*_{t+1}$$

  Why not?

PEA & RE

- Suppose η_n^* is the fixed point we are looking for
- So with η_n^* we get best predictor of y_{t+1}
- Does this mean that solution is a rational expectations equilibrium?

Disadvantages of stoch. sim. PEA

- The inverse of $X'X$ may be hard to calculate for higher-order approximations
- Regression points are clustered $\Longrightarrow$ low precission
  - recall that even equidistant nodes are not enough for uniform convergence; with simulated date, the "nodes" are even less spread out with stochastic PEA)

Disadvantages of stochastic PEA

- Projection step has sampling error
  - this disappears slowly (especially with serial correlation)

Advantages of simulated nodes

- Regression points are clustered

  $\Longrightarrow$ better fit *where it matters* **IF** functional form is poor

  (with good functional form it is better to spread out points)

Advantages of simulated nodes

- **BIG ADVANTAGE:** Not subject to the exponential curse of dimensionality as standard projection methods.

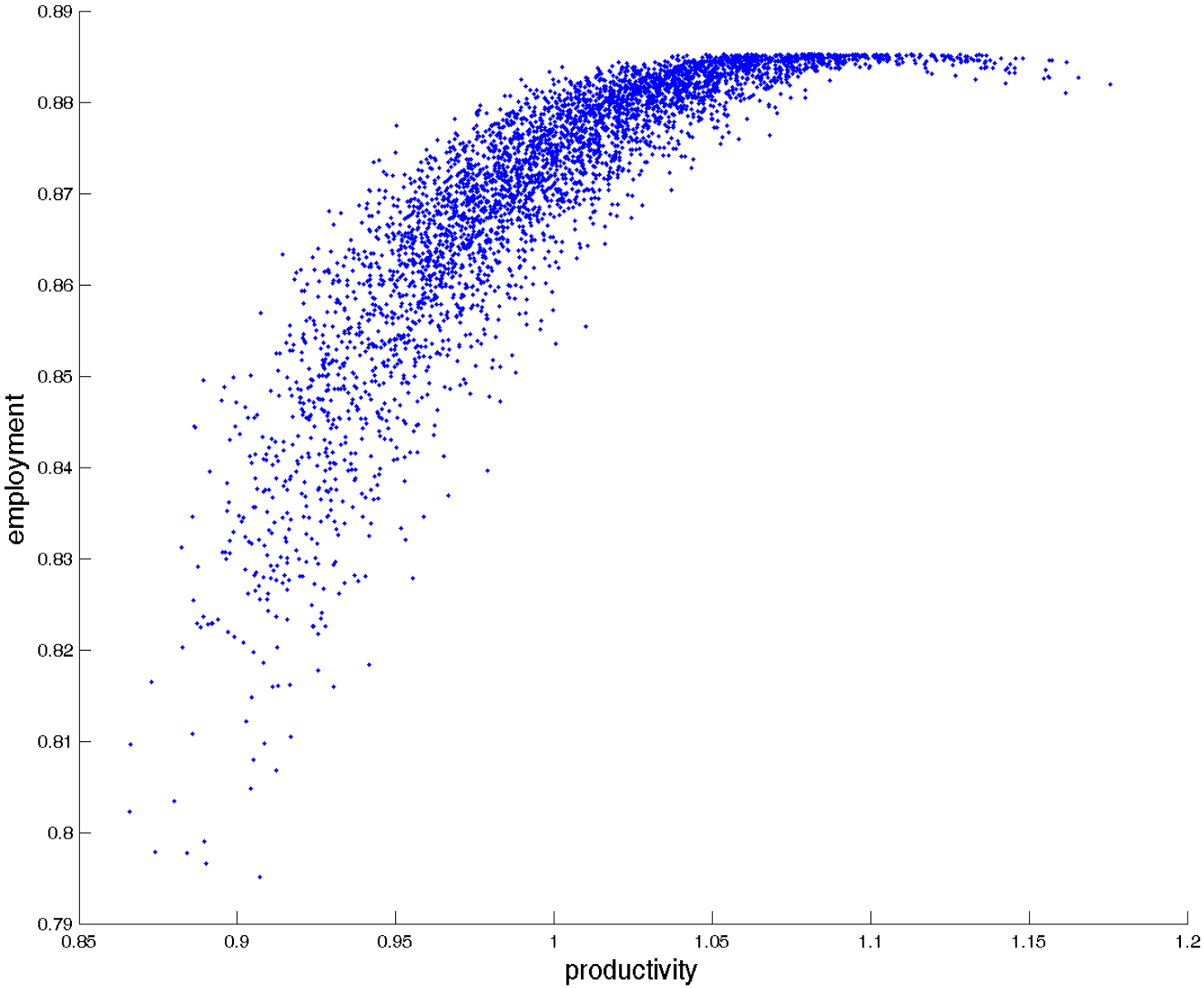
Advantages of simulated nodes

- Grid: you may include impossible points

  Simulation: model iself tells you which nodes to include

  - (approximation also important and away from fixed point you may still get in weird places of the state space)

Odd shapes ergodic set in matching model



Non-stochastic Simulations PEA

Improvements from Maliar, Maliar & Judd (2010,2011)

1. Use flexibility given to you
2. Use $\widehat{\mathsf{E}}[y_{t+1}]$ instead of y_{t+1} as regressand
   - $\widehat{\mathsf{E}}[y_{t+1}]$ is numerical approximation of $\mathsf{E}[y_{t+1}]$
   - even with poor approximation the results improve !!!
3. Improve regression step

Use flexibility

Many E[]'s to approximate.

1. Standard approach:

$$c_t^{-\nu} = \mathsf{E}_t \left[\beta c_{t+1}^{-v} \alpha \beta c_{t+1}^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

2. Alternative:

$$k_{t+1} = \mathsf{E}_t \left[k_{t+1} \beta \alpha \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\nu} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

- Such transformations can make computations easier, *but* can also affect stability of algorithm (for better or worse)

E[y] instead of y as regressor

- $\mathsf{E}[y_{t+1}] = \mathsf{E}[f(\varepsilon_{t+1})]$ with $\varepsilon_{t+1} \sim N(0, \sigma^2)$

  $\Longrightarrow$ Hermite Gaussian quadrature can be used

  (MMJ: using $\widehat{\mathsf{E}}[y_{t+1}]$ calculated using one node is better than using y_{t+1})

- Key thing to remember: sampling uncertainty is hard to get rid off

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} y_{t+1} &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + u_{t+1} \\ u_{t+1} &= \text{prediction error} \end{aligned}$$

- Then you **cannot** estimate coefficients using LS based on

$$\ln\left(y_{t+1}\right) = a_o + a_1 \ln k_t + a_2 \ln z_t + u^*_{t+1}$$

- You have to use non-linear least squares

E[y] instead of y as regressor

- Suppose:

$$\begin{aligned} \mathrm{E}\left[y_{t+1}\right] &= \exp\left(a_o + a_1 \ln k_t + a_2 \ln z_t\right) + \bar{u}_{t+1} \\ \bar{u}_{t+1} &= \text{numerical error} \end{aligned}$$

- Then you **can** estimate coefficients using LS based on

$$\ln \mathrm{E}\left[y_{t+1}\right] = a_o + a_1 \ln k_t + a_2 \ln z_t + \bar{u}^*_{t+1}$$

- Big practical advantage

Simple way to improve regression

- The main underlying problem is that $X'X$ is ill conditioned which makes it difficult to calculate $X'X$
- This problem is reduced by

1. Scaling so that each variable has zero mean and unit variance
2. Hermite polynomials

Hermite polynomials; Definition

$$P_n(x) = \sum_{j=0}^{n} a_j H_j(x)$$

where the basis functions, $H_j(x)$, satisfy

$$\mathrm{E}\left[H_i(x) H_j(x)\right] = 0 \text{ for } i \neq j$$

$$\text{if } x \sim N(0,1)$$

Hermite polynomials; Construction

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_{m+1}(x) &= xH_m(x) - mH_{m-1}(x) \text{ for } j > 1 \end{aligned}$$

This gives

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \\ H_5(x) &= x^5 - 10x_3 + 15x \end{aligned}$$

One tricky aspect about scaling

Suppose one of the explanatory variables is

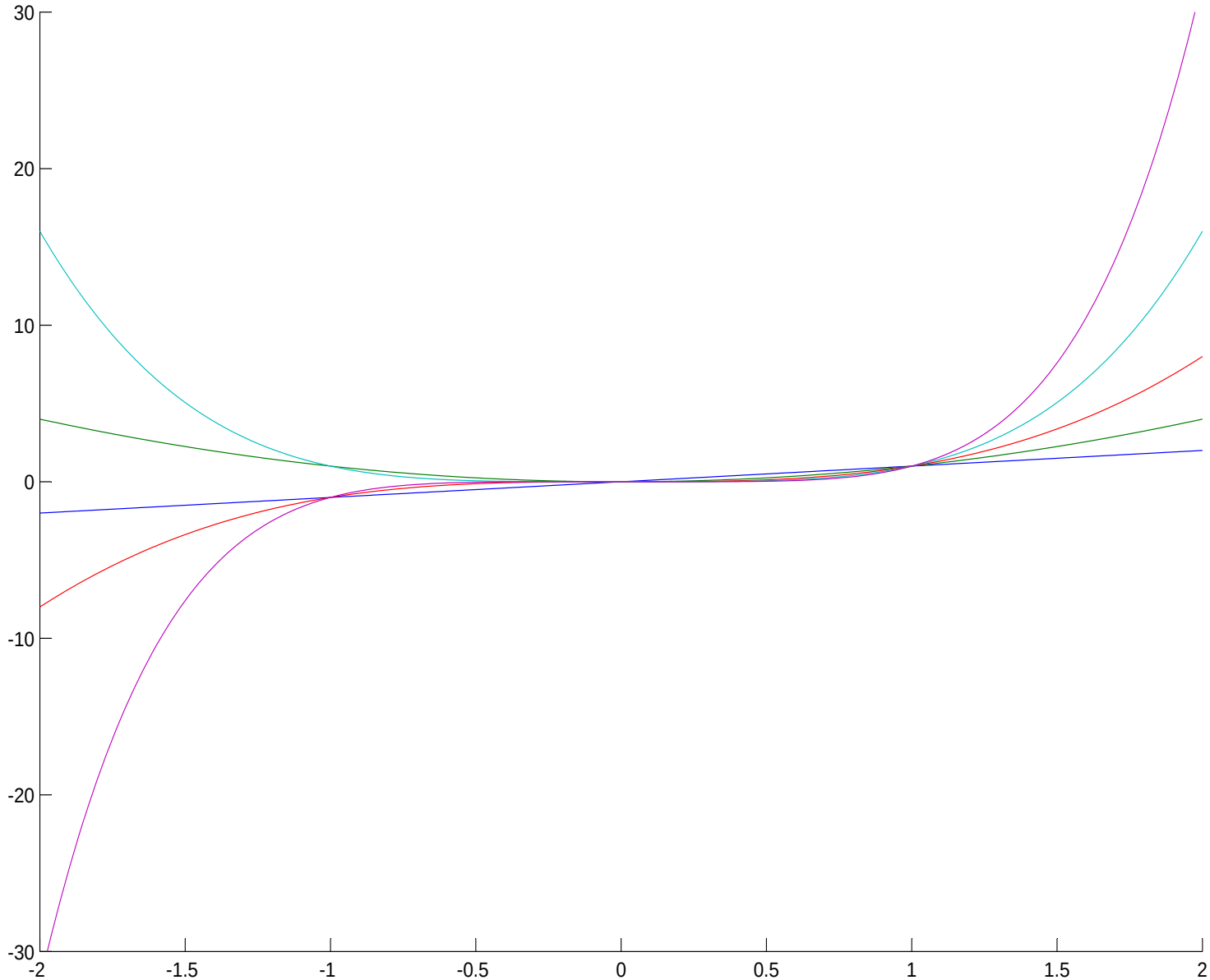
$$x_t = \frac{k_t - M_T}{S_T}$$

$$M_T = \sum_{t=1}^{T} k_t / T \;\&\; S_T = \left(\sum_{t=1}^{T} (k_t - M_T)^2 / T \right)^{1/2}$$

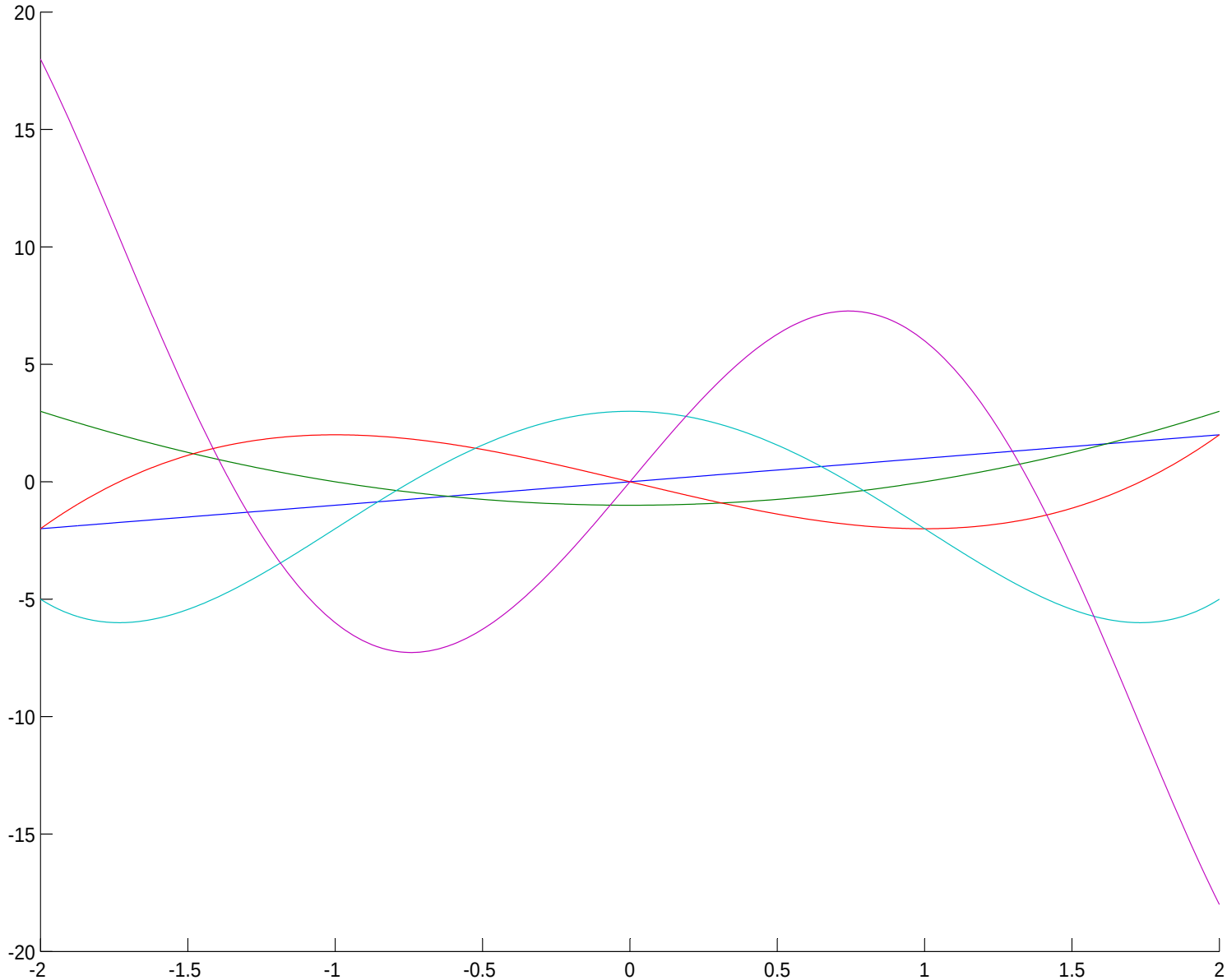
One tricky aspect about scaling

- $\Longrightarrow$ each iteration the explanatory variables change (since M and S change)
- $\Longrightarrow$ taking a weighted average of old and new coefficient is odd
- I found that convergence properties can be quite bad
- So better to keep M_T and S_T fixed across iterations

Two graphs say it all; regular polynomials



Two graphs say it all; Hermite polynomials



More ways to improve regression

1. LS-Singular Value Decomposition
2. Principal components

See Maliar, Maliar, Judd (2010,2011) for details.

PEA and learning

- Traditional algorithm:
  - simulate an economy using belief η_n^i
  - formulate new belief η_n^{i+1}
  - simulate *same* economy using belief η_n^{i+1}

PEA and learning

- Alternative algorithm to find *fixed point*
  - simulate T observations using belief η_n^{T-1}
  - formulate new belief η_n^T
  - generate 1 more observation
  - use $T+1$ observations to formulate new belief η^{T+1}
  - continue

  When convergence has taken place is more difficult to determine, since each additional observation has smaller weight since T increases.

PEA and learning

- Modification of alternative algorithm is economically interesting
  - simulate T observations using belief η_n^{T-1}
  - use τ observations to formulate new belief η_n^T
  - generate 1 more observation
  - use last τ observations to formulate new belief η^{T+1}
  - continue
- Beliefs are based on limited past $\Longrightarrow$ time-varying beliefs

PEA and learning

- Suppose the model has different regimes
  - e.g. high productivity and low productivity regime
  - agents do not observe regime$\Longrightarrow$ it makes sense to use limited number of past observations
- With the above algorithm agents gradually learn new law of motion

References

- Den Haan, W.J. and A. Marcet, 1990, Solving the stochastic growth model with parameterized expectations, Journal of Business and Economic Statistics.
- Den Haan, W.J., Parameterized expectations, lecture notes.
- Heer, B., and A. Maussner, 2009, Dynamic General Equilibrium Modeling.
- Judd, K. L. Maliar, and S. Maliar, 2011, One-node quadrature beats Monte Carlo: A generlized stochastic simulation algorithm, NBER WP 16708
- Judd, K. L. Maliar, and S. Maliar, 2010, Numerically stable stochastic methods for solving dynamics models, NBER WP 15296